„A STUDY OF COLLECTIVE COORDINATES AND DYNAMICAL GROUPS IN NUCLEAR THEORY"

„Untersuchungen von kollektiven Koordinaten und dynamischen Gruppen in der Theoretischen Kernphysik"

Dissertation

zur Erlangung des Grades eines Doktors
der Naturwissenschaften

der Fakultät für Physik
der Eberhard-Karls-Universität zu Tübingen

vorgelegt von

ZORKA PAPADOPOLOS

aus Beograd

1983

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A b s t r a c t

PAPADOPOLOS, ZORKA:

A STUDY OF COLLECTIVE COORDINATES AND DYNAMICAL GROUPS IN NUCLEAR THEORY

Lie-algebraic techniques for the group action on manifolds given as a direct product of coset spaces and group manifolds are developed.

The microscopic realisation of the Mass Quadrupole Collective Model (MQC) in the $\mathfrak{SO}(3)\times SO(n)$ and $\mathfrak{gl}_+(3,\mathbb{R})\times SO(n)$ schemes is studied. The problem of the separation of the kinetic energy and the velocity field into a collective and an intrinsic part is analyzed.

Different coordinate schemes in phase space for the U(n)-invariant collective motion and the $\mathfrak{U}(3)$ dynamical group are introduced. In the $\mathfrak{gl}(3,\mathbb{C})\times U(n)$ scheme, the invariant volume element in the new coordinates and a completely orthonormal basis is constructed.

UNTERSUCHUNGEN VON KOLLEKTIVEN KOORDINATEN UND DYNAMISCHEN GRUPPEN IN THEORETISCHER KERNPHYSIK

Lie-algebraische Techniken für Gruppenwirkungen auf Mannigfaltigkeiten, die als direkte Produkte von Nebenklassenräumen und Gruppenmannigfaltigkeiten gegeben sind, werden entwickelt.

Die mikroskopische Realisierung des "Mass Quadrupole Collective Model" (MQC) in den $\mathfrak{SO}(3)\times SO(n)$ und $\mathfrak{gl}_+(3,\mathbb{R})\times SO(n)$ Schemata wird untersucht. Die Separation der kinetischen Energie und des Geschwindigkeitsfeldes in einen kollektiven und einen intrinsischen Anteil wird analysiert.

Verschiedene Koordinatenschemata im Phasenraum für U(n)-invariante kollektive Bewegung und die dynamische Gruppe $\mathfrak{U}(3)$ werden eingeführt. Im $\mathfrak{gl}(3,\mathbb{C})\times U(n)$ Schema werden das invariante Volumenelement in den neuen Koordinaten und eine vollständige Orthonormalbasis konstruiert.

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General introduction

Bohr and Mottelson[1,2] (1952,1953) introduced their phenomenological model for collective motion in which incompressible nuclear matter moves irrotationally. The kind of motions studied were rotations and small vibrations. Coordinates for that motion are Euler angles (φ, θ, ψ) for rotations and β and γ coordinates for quadrupole deformations and vibrations. The mean radius R_o is fixed by the constant volume assumption. The coupling between the total angular momentum of the system (acting on Euler angles) and the angular momentum $\vec{j}$ of the valence particles not included in the core was given by the Coriolis term in the kinetic energy. The Bohr and Mottelson model (BM) was successful in a simple explanation of certain basic features of nuclear collective motion (for example the non-rigid moments of inertia), but it was too restrictive in its assumption to encompass even in a phenomenological framework the variety of nuclear collective phenomena.

As in these times the shell-model theory for single particle motion was successfully developed, it was natural that many theories for nuclear collective motion appeared based on some modifications of the shell-model theory. Collective effects were reformulated in terms of the self-consistent H F theory and its better versions HFB and RPA. These are theories which introduce the residual interaction w.r. t. the self-consistent field approximation. The author of this thesis is not a specialist in these methods so let the monograph by Ring and Schuck[3] be mentioned as a general reference. All these theories tried to organize collective spectra and to separate them from the single particle effects. Rotational and vibrational spectra were studied, their separation and mixing was analyzed. The problems of stable deformations, effective moments of inertia and the backbending (see for example ref. 4) have also been studied with the inclusion of the effects of pairing. One examined monopole excitations, giant resonances, problems of retaining fundamental symmetries, which had in many cases

been solved through some projection techniques before or after variation , and so on. The big advantage of all these theories based on the shell-model compared to the phenomenological BM theory is that they are microscopic and the Pauli principle is automatically included through the fermion character of the particles. The disadvantage is that these models and corresponding approximations are formulated in a state space based on single-particle states and hence do not easily allow for a simple geometric analysis of the problem, as was the case in the BM model.

Independently of the shell-model theory, many group theoretical methods have been applied and developed to cope with the problems of collective motion. Well-known is the concept of a symmetry group, which leads to a degenerate spectrum. Two other important concepts in the frame of group theoretical methods have been introduced in order to solve the problem of collective motion: (1) the concept of dynamical groups which should organize the collective spectra (the concept of dynamical compact groups was first introduced by Gell-Mann in elementary particle physics) and (2) the concept of a coordinate transformation of 3A coordinates (A is the number of nucleons) to some other set in which coordinates related to the collective motion appear. Those coordinates represent some generalization of the BM collective coordinates. The third very important, geometric concept of kinematical groups appears first implicitly and later explicitly in relation to both concepts.

We give first some brief review of the historical development of the concept of dynamical groups. Elliott (1958)[5] was the first to introduce a dynamical group in nuclear physics. His unitary unimodular compact group SU(3) "organized" the 2s-1d-shell rotational spectra into finite bands. These bands belong to the irreducible representations (irreps) of the SU(3) group. That is clearly an alternative method to the methods based on a single deformed state chosen for example to be a Slater determinant of anisotropic harmonic oscillator states[6,7] or some other deformed single particle basis[8], or two-cluster states[9], and so on. This deformed state is then projected to fixed angular momenta[10] in order to obtain

the rotational bands. We will try to make the concept of a dynamical group clear through the example of SU(3). The hamiltonian for the SU(3) model is given as $H = H_s + H_{int}$. The part of the hamiltonian H_s has the group SU(3) as the symmetry group and its spectrum therefore is degenerate within an irrep of SU(3). The part H_{int} violates the SU(3) symmetry and is not small w.r. t. H_s, as it would be in some conventional perturbation theory. The most important feature of H_{int} is that it must be some function of the generators of the dynamical group SU(3). The spectrum of H is now organized into families of rotational bands distinguished from each other by the labels $(\lambda\mu)$ of SU(3) irreps. If one can find an expression for the transition operator within the algebra of the dynamical group, it connects the states only within one family of states.

The next step was the introduction of spectrum generating algebras (SGA)[11], which are always related to noncompact groups. In the broader sense these are also dynamical groups. Unitary irreps of the noncompact groups are, as is well-known, infinite dimensional. An SGA should be such that a few infinite irreps supply a basis for all the eigenstates of H which are of physical interest. Let us give some simple illuminating example: the set of all states of an N-dimensional harmonic oscillator belongs to one of the two irreps of the group $Sp(2N, \mathbb{R})$[12]. An SGA for the collective model was first introduced and extensively studied by Weaver, Biedenharn and Cusson[13,14,15]. They introduced in their model the unimodular linear group $\mathfrak{SL}(3,\mathbb{R})$. An action of $\mathfrak{SL}(3,\mathbb{R})$ on $\mathbb{R}^{3A}$, the configuration space for A particles, yields the most general linear collective point transformation conserving the volume element. This action of $\mathfrak{SL}(3,\mathbb{R})$ on $\mathbb{R}^{3A}$ implements on the mass quadrupole tensor

$$Q_{ij} := \sum_{s=1}^{A} \xi_{is}\xi_{js} - \frac{1}{3}\sum_{k=1}^{3}\sum_{s=1}\xi_{ks}\xi_{ks}\delta_{ij} \; ; \quad \xi_{is} \in \mathbb{R}^{3A} \quad i=1,2,3$$

an action in the following way

$$g_{ik}\xi_{ks} = \hat{\xi}_{is} \; , \qquad \hat{Q}_{ij} := \sum_{k,l} g_{ik}g_{jl}Q_{kl} \; , \quad g \in \mathfrak{SL}(3,\mathbb{R}) .$$

This use of the $\mathfrak{SL}(3, \mathbb{R})$ is an example of a kinematical group.

i.e. a group which yields a geometric transformation of certain (collective) coordinates of the system. The kinematical group $SL(3,\mathbb{R})$ brings very important changes compared to the BM theory, it introduces vorticities and their momenta, vortex spin which is the generalization of the BM irrotational flow. The introduction of vorticities is analogous to the introduction of pairing correlations[13], it leads to the effective moment of inertia by rotational motion corrected w.r. t. the one which corresponds to the irrotational flow. The SGA of the authors in ref. 14 and 15 is $cm(3)$ with eight generators from $SL(3,\mathbb{R})$ plus five Q_{ij} generators plus the additional operator $\sum_{i=1}^{3}\sum_{s=1}^{m} \xi_{is}^2$. The authors particularly studied the representations of the contraction of the $cm(3)$ algebra related to the group $T_5 \circledast SO(3)$, the semidirect product of the abelian group T_5, the translation group in 5 dimensions (whose generators are 5 Q_{ij}'s), and SO(3). This contraction physically means that the energy excitations coming from the deformation of Q_{ij} are pushed to infinity, and the group so obtained provided them with the rotational spectra only. This group was generalized to $T_5 \circledast SU(2)$ to include the spin. Its representations are infinite rotational bands. These authors also studied the contact transformation based on the $SL(3,\mathbb{R})$ point transformation, obtained the classical collective kinetic energy as the sum of $T_{vibrational} + T_{rotational}$ and then quantized it.

Another group of authors, Rowe, Rosensteel and collaborators[16,17] extended the $cm(3)$ model by including the generators for the monopol mode. These authors have for the kinematical group the general linear group $gL_+(3,\mathbb{R})$ which does not conserve volume, and instead of the traceless mass quadrupole tensor they introduce $Q_{ij} = \sum_{s=1}^{m} \xi_{is}\,\xi_{js}$ which has 6 independent components. These 6 components with 9 $gL_+(3,\mathbb{R})$ generators close and give the 15 dimensional MQC algebra.

All these models for collective motion based on SGA and kinematical groups $(SO(3), SL(3,\mathbb{R}), gL_+(3,\mathbb{R}))$ of the mass quadrupole tensor are phenomenological. The important problem appeared: how to link

these models to the many-particle dynamics.

We already mentioned that in the BM theory collective coordinates appear. For any microscopic theory one starts from single particle coordinates. The clear question appeared how to combine 5,6 or more collective coordinates with 3A single particle coordinates so that collective coordinates appear as some function of 3A single particle coordinates and still to have at disposal the single particle coordinates, or at least some relevant information out of them necessary for the microscopic theory. Different authors tried very early to solve this problem such as Lipkin et al (1955)[18] and Villars (1957)[19,20]. They introduced some method of redundant coordinates. They "added" collective coordinates to single particle coordinates and introducing simultaneously holonomic constraints (which imposed that the mass quadrupole tensor should be diagonal in the body-fixed frame) they managed to cope with a 3A-dimensional problem. They claimed that it was important to keep both collective coordinates and single particle coordinates in order to combine geometric and shell-model aspects of the problem. It should be mentioned that already in the analysis of both authors there appeared, apart from the angular momenta of the whole system, some other angular momenta in the body as a pure consequence of the coordinate transformations, which were later identified as vortex spin. For both authors the kinematical group is $SO(3)$.

Much success in coordinate transformations in the theory of collective motion was achieved introducing the SO(n) group to "organize" intrinsic coordinates. This was done first by Zickendraht[21] and then by Dzyublik et al[22] independently, and especially nicely and clearly by Buck, Biedenharn and Cusson[23]. All these authors had both the kinematical group $SO(3)$ and SO(n) group in their analyses (n = A-1, center-of-mass being separated), but Rowe and Rosensteel[24] were the first to formulate explicitly an orbit analysis. They consider it as a $g\mathcal{L}_+(3,\mathbb{R}) \times SO(n)$ scheme. Vanagas gave a meaning to the SO(n) group by postulating O(n) or SO(n) invariance for the

collective hamiltonian in orbital space. The group S_{n+1} of orbital permutations can be embedded in the group O(n) and therefore the approach of Vanagas can be seen as an extension of the supermultiplet approximation. The spin and isospin can then be treated within this scheme. As the result of all these coordinate transformations[21,22,23,24,25] the kinetic energy of A particles so transformed appeared as a polynomial expression in 12 out of 15 generators of MQC, interpreted as its collective part[23,24,25], plus an expression quadratic in 3n-9 intrinsic generators. Whether this result can be really interpreted as the complete separation w.r. t. $gl_+(3,\mathbb{R})$ collectivity as was done in ref. 23,24,25 and 26 will be discussed in chapter II and III. It should also be said that, although vibrational coordinates and momenta are present in MQC, the basis for the vibrational spectra has not yet been found except for the 1-dimensional case.

One more historical source of particular interest for the present work should be mentioned. In contrast to Elliott, whose SU(3) group was referred to the state space in the 2s-1d shell, Kramer and Moshinsky[27] introduced the U(3)xU(A), geometrical group, as a physically relevant subgroup of U(3A). U(3A) is the symmetry group of the 3A-dimensional oscillator related to A particles in a 3-dimensional harmonic oscillator potential. They studied among other topics the quadrupole-quadrupole interaction as an approximation for the Gaussian two-body interactions for multishell configuration and also for a single shell and obtained the rotational bands. It is again an example for the dynamical group similar to the one of Elliott but having additional geometric meaning.

Having given some historical background of how the ideas in group theoretical models for collective motion were developed, we are now going to describe the present way of looking upon them, or rather the way we see these problems and cope with them. Orbit analysis appears as an important term in our work. We will try to explain it here in a nontechnical way.

Consider a manifold $\mathcal{M}$ and an action of a group $\widetilde{\mathcal{G}}$ on $\mathcal{M}$. This action is called transitive if any pair of points from $\mathcal{M}$ can be taken as an original and image point under the mapping $\widetilde{\mathcal{G}}\times\mathcal{M}\to\mathcal{M}$. A non-transitive manifold can be split under the group action into transitive submanifolds which are called the orbits on $\mathcal{M}$ under $\widetilde{\mathcal{G}}$. An orbit on $\mathcal{M}$ may be characterized by a single representative point together with all its image points under $\widetilde{\mathcal{G}}$. A collection of representative points, one from each orbit, is called a transversal w.r. t. the group action. In all cases to be considered, the transversal will be a smooth submanifold. The combination of all the orbits plus the transversal clearly spans the manifold $\mathcal{M}$.
In this work the manifold $\mathcal{M}$ in question in chapter II and III is the Euclidean space $\mathbb{R}^{3A}$ for A nucleons in configuration space and the group action $\widetilde{\mathcal{G}}$ is the linear point transformation. In chapter IV and V our manifold is phase space for A particles, or rather its equivalent complex realization. The groups w.r. t. which we are going to do our orbit analysis in configuration space are $SO(3)\times SO(n)$ and $\mathcal{GL}_+(3,\mathbb{R})\times SO(n)$. The group actions which we study in phase space are $U(3)\times U(n)$ and $\mathcal{GL}(3,\mathbb{C})\times U(n)$, and they are all linear canonical transformations and therefore subgroups of the symplectic group $Sp(6A,\mathbb{R})$ or more precisely, its isomorphic complex realization $Sp(6A,\mathbb{C})\cap U(n,n)$. $\mathcal{SO}(3)$ and $\mathcal{GL}_+(3,\mathbb{R})$ are kinematical groups, and O(n) and U(n) dynamical groups and we call all of them collective groups. O(n) and U(n) are symmetry groups which define the collective hamiltonian as O(n) and U(n) invariant respectively[28,29], and we call them intrinsic groups. We denote collective groups by script letters and intrinsic groups by latin letters. The point in configuration space is given by Cartesian coordinates organized in 3xn matrices $\xi=(\xi_{is})$ whose first label i is the space label i = 1,2,3 and s is the particle label s = 1,2,...,A. The natural action of the group $\widetilde{\mathcal{G}}$ is such that the collective group acts on all particles in the same way which is realized by the action on the label "i" as the left action, andthe intrinsic group acts on label s as the right action. Both actions clearly commute. In phase space the same holds true except that the point is a

6xA real matrix or equivalently 3xA complex matrix. From the concept of an orbit analysis it is clear that each point in the manifold "sits" on some orbit, so if we take anyone and act on it with all group elements of our direct product group, we obtain all point on an orbit. If one does this for all points on a transversal one spans the whole manifold through the group action. How one chooses the transversal is the technical problem to be solved in each particular case.

So far, any point on the manifold can be parametrized by some freely chosen coordinates of a certain point on the transversal and some group parameters of the acting group which determine a certain point on the orbit. But it almost always happens that the number of all these coordinates and parameters is bigger than the dimension of the manifold. So some coordinates become redundant and one should search for a certain number of independent constraints in order to reduce the number of coordinates to the dimension of the manifold. For that purpose the well-defined method of a stability group H for different types (if they exist) of representative points on the transversal exists. The stability group is the subgroup of the acting group such that under its action the representative point is left uncharged. So we remove from the set of coordinates those parameters which characterize different elements of the stability group. By this procedure, each orbit is characterized by the group parameters of G minus the group parameters of H, or in other words by the coset which we denote as G/H .

Up to now we have not introduced a clear definition of which coordinates should be considered as collective coordinates and which as intrinsic coordinates. In the configuration space orbit analysis different points of view are possible. One of them is to consider that the $GL_+(3,R)$ coordinates are collective. This concept introduces as collective effects rotations, deformations, vibrations and vorticities.Which coordinates, or rather, which submanifold should be considered as intrinsic is hard to tell. One could say: the manifold spanned by all other coordinates. We found that this problem is not yet solved and we discuss

it related to the problem of separation of the kinetic energy into the collective and intrinsic part in chapter II and III particularly. The other criterion, proposed by Vanagas, is to consider all parameters which change under the right SO(n) action as intrinsic coordinates. This clearly can be seen as a statement in the $SO(3)\times SO(n)$ scheme. It yields that collective coordinates can then be chosen on the transversal to the SO(n) action. Since the orbital permutations are embedded in the group $O(n, \mathbb{R})$, the collective coordinates are automatically unchanged under permutations. The transversal to the SO(n) action is spanned by the $SO(3)$ orbits (3 rotational coordinates) and by the transversal to the $SO(3)\times SO(n)$ action (3 coordinates for vibrations and deformations). Note that this criterion of collectivity defined on the manifold is not identical to the one which demands O(n) invariance of the collective potential energy.
Let us summarize this discussion. Two concepts of collectivity in configuration space have been proposed: (I) collective coordinates are all parameters of the collective group and they are natural coordinates for the orbits of the collective group alone; (II) collective coordinates span the transversal to the right action of the intrinsic group alone. Both criteria we also state for the orbit analysis in phase space.

So far we have not distinguished between an orbit analysis in phase space and in the configuration space. In both cases our acting group is the direct product of two groups, a collective and an intrinsic one. Now we want to point out crucial differences. We already mentioned that in the configuration space one introduces a kinematical group, for example $gl_+(3,\mathbb{R})$. Then one does a coordinate transformation related to the $gl_+(3,\mathbb{R})\times SO(n)$ orbit analysis. Then one calculates the Jacobi matrix and its inverse in order to be able to write the old momenta as a superposition of new momenta canonically conjugate to new collective and intrinsic coordinates. Thus one obtains the contact transformation in phase space induced through the point transformation in configuration space. Of phyiscal interest for

collective theory are some other momenta: the angular momenta $L_{ij}(i<j = 1,2,3)$, the vibrational momenta $t_i(i = 1,2,3)$ and vortex spin $\alpha_{ij}(i<j = 1,2,3)$. This momenta can be written as linear combinations of the canonically conjugate collective momenta with coefficients depending on the collective coordinates. Together with 6 quadrupole collective coordinates $Q_{ij}(i\leq j = 1,2,3)$ after the quantization they form the dynamical group MQC. If the orbit analysis is applied to the full phase space, it allows for a completely different procedure. In the phase space we introduce the collective group to be some subgroup of $Sp(6,\mathbb{R})$(note that MQC$< Sp(6,\mathbb{R})$, also!). This group is both kinematical and dynamical. It is dynamical in the sence that we expect that it generates the spectra for the collective motion. It is chosen to be kinematical also in the sence that it introduces together with the intrinsic group a coordinate transformation in phase space which is in general nonlinear. We suppressed here the concept of the intrinsic group which gives us the link to the microscopic theory. As we already said, Vanagas proposed the O(n) invariance as an ansatz for the collective hamiltonian. This ansatz states in phase space[29,30] via complementarity[31] (which is a technical term in representation theory explained in (IV.0)) gives a link to the Sp(6, $\mathbb{R}$) dynamical group. The group Sp(6, $\mathbb{R}$) or rather Sp(2r, $\mathbb{R}$) (r = 1,2,3 is the dimension of the geometric space) as a phenomenological group for collective motion was proposed by Goshen and Lipkin for r = 1[32] and 2[33] and broadly studied by Rosensteel and Rowe[34] for the three dimensional space (r = 3). That group explained collective vibrations for the first time. It also has the MQC and SU(3) as its subgroups. In ref. 30 for the group pair $Sp(6,\mathbb{R})\times O(n)$ the orbit analysis was done. In this work we study the $U(3)\times U(n)$ pair. The orbit analysis for this pair of groups was the first orbit analysis in phase space related to collective motion[35].

Now we give the short review of the content of this work. In chapter I we define the group action on a group manifold or coset space. We study both right and left actions, coordinate independent and coordinate dependent for the orthogonal and unitary groups. This chapter provides us with the necessary

tools for what we refer to as our method which was briefly present here: 1. Choose the collective and intrinsic group and do the orbit analysis. 2. Find the Jacobi matrix and invert it (here we mention that the other authors work only with the metric tensor which is useful only for the transformation of the expression for kinetic energy) in order to find the new vector fields and momenta which appear in all expressions for energy, velocity fields and so on. 3. Find the left and the right action generators for the acting group $\tilde{G}$. 4. Find the invariant volume element in new coordinates using the Jacobi matrix. 5. Construct the basis for your problem in state space.

In chapter II we apply our method to the problem already solved in the literature[21,22,23,24], the $SO(3)\times SO(n)$ scheme. We claim that this result cannot be discussed from the point of view of the $gl_+(3,\mathbb{R})$ collectivity.

In chapter III we do exactly the program that a lot of authors have stated but never have done: we analyse the $gl_+(3,\mathbb{R})\times SO(n)$ scheme and claim that it is rather different from the $SO(3)\times SO(n)$ scheme. In this scheme it becomes clear that the separation of the collective kinetic energy and the velocity field is possible only under certain constraints which we obtain and do not postulate.

In chapter IV we develop the $U(3)\times U(n)$ scheme w.r.t. both concepts of collectivity.

In chapter V we introduce $gl(3,\mathbb{C})$ collectivity via the $gl(3,\mathbb{C})\times U(n)$ scheme which provides us with one orbit which includes all the orbits of the $U(3)\times U(n)$ scheme. $gl(3,\mathbb{C})$ collectivity also includes both concepts of collectivity of chapter IV. In this case we are able to construct the basis from products of pure collective and intrinsic states.

I. Group action on groups and cosets

I.0 Introduction

In our analysis the final goal should be the extraction of the collective dynamics out of some microscopic theory. Motivated by the physics of our problem we introduce collective and intrinsic groups. Through the coordinate transformation and orbit analysis without any ad hoc restrictions we incorporate the group parameters as a subset of coordinates in configuration or phase space manifold for A particles and extract the part of the manifold where we hope that collective dynamics should take place. On this collective manifold one should write the expressions for all physical observables as energy, velocity field and so on. In order to construct different vector fields and momenta which appear in all these expressions in new coordinates, we have to invert analytically the Jacobi matrix for the coordinate transformation which is an 3nx3n real (or complex) matrix. The only reason that we are sometimes able to solve such a complicated problem lies in the fact that the larger part of our new coordinates are parameters of some group manifolds or coset spaces and in this way provide us with certain blocks in the Jacobi matrix which we can partially invert. A similar problem appears in the determination of invariant volume element. The content of this chapter provides us with some general and particular knowledge about the left and right action of $\widetilde{G}$ on the orbit manifolds which are either group manifolds themselves, or certain coset spaces.

I.1. The action of a Lie group on its parameter space

An action of a group $\tilde{G}$ on some manifold $\mathcal{M}$ can be defined. In some cases it makes sense to introduce the "left" or "right" action of some group on a certain manifold. For example if $\tilde{G}$ is $SO(3)\times S_N$ where $SO(3)$ represents simultaneous rotations in configuration space of N particles, S_N is the group of permutations of N particles and $\mathcal{M}$ is the $\mathbb{R}^{3N}$ manifold of 3 x N matrices $\xi=(\xi_{is})$ where i = 1,2,3 is the space index and s = 1,2,...,N is the particle index, then the "natural" action is defined through $SO(3)$ left action and S_N right action on the indices i and s respectively. We shall give the generators for the left and the right action for certain examples in (II.3)* and (IV.3). In this section we study only the special case where the manifold on which the group acts is chosen to be the group manifold itself. So we study the left and the right action of $\tilde{G}$ on G. From now on "$\sim$" always marks the acting group.

I.1.1 Definition: The left action of $\tilde{G}$ on some scalar function of G is defined by

$$\left({}^{L}T_{\tilde{g}}F\right)(g) = F(\tilde{g}^{-1}g) = F\left(\Phi(\tilde{\gamma}^{-1},\chi)\right)$$

$\tilde{g}(\tilde{\chi})\in\tilde{G}$, $g(\chi)\in G$, Φ is the structure function of the group (both, $\tilde{G}$ and G)

$$\chi'_{\mu} = \Phi_{\mu}(\tilde{\gamma}^{-1},\chi),$$

where we introduce the convention that for group elements inverse to each other the corresponding parameters are γ and γ^{-1} respectively. (Clearly in the case of exponential parametrization $\gamma^{-1}\equiv-\gamma$).

I.1.2 Definition: Given the structure function $\Phi_{\mu}(\gamma,\chi)$ of the group, the matrix ${}^{L}\Theta(\chi)$ can be introduced,

$${}^{L}\Theta_{\rho\mu}(\chi) := \frac{\partial}{\partial\tilde{\gamma}_{\rho}}\Phi_{\mu}(\tilde{\gamma},\chi)\Big|_{\tilde{\gamma}=0}.$$

*(II.3) means: chapter II section 3. From now on we are going to use this short-hand notation.

I.1.3 Definition: The generators for the left action $\tilde{G} \times G \to G$ are

$$ {}^{L}X_{\varrho}(x) = - \sum_{\mu} {}^{L}\Theta_{\varrho\mu}(x) \frac{\partial}{\partial x_{\mu}} . $$

One introduces them in a natural way:

$$ x'_{\mu} = \Phi_{\mu}(\delta\tilde{\gamma}^{-1}, x) \approx \Phi_{\mu}(0,x) + \sum_{\varrho} \delta\tilde{\gamma}_{\varrho}^{-1} \left. \frac{\partial \Phi_{\mu}(\tilde{\gamma}^{-1}, x)}{\partial (\tilde{\gamma}^{-1})_{\varrho}} \right|_{\tilde{\gamma}^{-1}=0} $$

$$ = \Phi_{\mu}(0,x) - \sum_{\varrho} \delta\tilde{\gamma}_{\varrho} \left. \frac{\partial \Phi_{\mu}(\tilde{\beta}, x)}{\partial \tilde{\beta}_{\varrho}} \right|_{\tilde{\beta}=0} $$

$$ = \Phi_{\mu}(0,x) - \sum_{\varrho} \delta\tilde{\gamma}_{\varrho} {}^{L}\Theta_{\varrho\mu}(x) \qquad (I.1.2)^{*} $$

$$ = x_{\mu} + \delta x_{\mu} . $$

$$ F'(x) := F(x') \approx F(x + \delta x) $$

$$ = F(x) + \sum_{\mu} \delta x_{\mu} \frac{\partial}{\partial x'_{\mu}} F(x') \Big|_{x'=x} $$

$$ = F(x) + \sum_{\varrho} \delta\tilde{\gamma}_{\varrho} \left[- \sum_{\mu} {}^{L}\Theta_{\varrho\mu}(x) \frac{\partial}{\partial x'_{\mu}} \right] F(x') \Big|_{x'=x} $$

$$ = F(x) + \sum_{\varrho} \delta\tilde{\gamma}_{\varrho} {}^{L}X_{\varrho}(x) F(x) . $$

I.1.4 Definition: The matrix ${}^{L}\Psi(x)$ ** is introduced as the inverse of ${}^{L}\Theta(x)$.

I.1.5 Definition: The right action of $\tilde{G}$ on some scalar function of G is defined

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** See ref. 36 p. 97 relation (2.6)

* The number (I.1.2) marked by * indicates the use of definition I.1.2. From now on we are going to use this short-hand notation.

$$\left({}^{R}T_{\tilde{g}}F\right)(g) = F(g\tilde{g}) = F\left(\Phi(x,\tilde{\gamma})\right)$$

$\tilde{g}(\tilde{\gamma})\in\tilde{G}$, $g(x)\in G$, Φ is the structure function of the group (both, $\tilde{G}$ and G)

$$x'_{\lambda} = \Phi_{\lambda}(x,\tilde{\gamma}).$$

I.1.6 Definition: Given the structure function $\Phi_{\mu}(x,\tilde{\gamma})$, the matrix ${}^{R}\Theta(x)$ can be introduced,

$${}^{R}\Theta_{\rho\mu}(x) := \frac{\partial}{\partial\tilde{\gamma}_{\rho}}\Phi_{\mu}(x,\tilde{\gamma})\Big|_{\tilde{\gamma}=0}.$$

I.1.7 Definition: The generators for the right action $G\times\tilde{G}\to G$ are

$${}^{R}X_{\rho}(x) = \sum_{\mu}{}^{R}\Theta_{\rho\mu}(x)\frac{\partial}{\partial x_{\mu}}.$$

One introduces them in a natural way:

$$x'_{\mu} = \Phi_{\mu}(x,\tilde{\gamma}) \approx \Phi_{\mu}(x,0) + \sum_{\rho}\delta\tilde{\gamma}_{\rho}\,\frac{\partial\,\Phi_{\mu}(x,\tilde{\gamma})}{\partial\,\tilde{\gamma}_{\rho}}\Big|_{\tilde{\gamma}=0}$$

$$= \Phi_{\mu}(x,0) + \sum_{\rho}\delta\tilde{\gamma}_{\rho}\;{}^{R}\Theta_{\rho\mu}(x) \qquad \text{(I.1.6)}$$

$$= x_{\mu} + \delta x_{\mu}.$$

$$F'(x) := F(x') \approx F(x+\delta x)$$

$$= F(x) + \sum_{\mu}\delta x_{\mu}\frac{\partial}{\partial x'_{\mu}}F(x')\Big|_{x'=x}$$

$$= F(x) + \sum_{\rho}\delta\tilde{\gamma}_{\rho}\left[\sum_{\mu}{}^{R}\Theta_{\rho\mu}(x)\frac{\partial}{\partial x'_{\mu}}\right]F(x')\Big|_{x'=x}$$

$$=: F(x) + \sum_{\rho}\delta\tilde{\gamma}_{\rho}\,{}^{R}X_{\rho}(x)\,F(x).$$

I.1.8 Definition: The matrix ${}^{R}\Psi(x)$ is introduced as the inverse of ${}^{R}\Theta(x)$.

Let us now look for the relation between right and left action.

I.1.9 Definition: We define adjoint matrix

$$ {}^{t}d_{\mu\rho}(x) = \left. \frac{\partial \Phi_\rho\left(\Phi(x,\gamma), x^{-1}\right)}{\partial \gamma_\mu} \right|_{\gamma=0} . $$

I.1.10 Proposition: The matrices $d(x)$ form a representation of G called the adjoint representation.

Proof:

$$ {}^{t}d_{\mu\rho}(x\psi) = \left. \frac{\partial \Phi_\rho\left(x\psi, \Phi(\gamma, (x\psi)^{-1})\right)}{\partial \gamma_\mu} \right|_{\gamma=0} . $$

We introduce here a short-hand notation and write $(x\psi)_\nu$ for $\Phi_\nu(x,\psi)$.

$$ {}^{t}d_{\mu\rho}(x\psi) = \sum_\sigma \left. \frac{\partial \Phi_\rho(x, \Phi(\delta, x^{-1}))}{\partial \delta_\sigma} \frac{\partial \Phi_\sigma(\psi, \Phi(\gamma, \psi^{-1}))}{\partial \gamma_\mu} \right|_{\substack{\delta = \psi\gamma\psi^{-1} \\ \gamma=0}} $$

$$ = \sum_\sigma \left. \frac{\partial \Phi_\sigma(\psi, \Phi(\gamma, \psi^{-1}))}{\partial \gamma_\mu} \right|_{\gamma=0} \left. \frac{\partial \Phi_\rho(x, \Phi(\delta, x^{-1}))}{\partial \delta_\sigma} \right|_{\delta=0} $$

$$ = \sum_\sigma {}^{t}d_{\mu\sigma}(\psi)\, {}^{t}d_{\sigma\rho}(x) . $$

Clearly it can be written

$$ d_{\mu\rho}(x\psi) = \sum_\sigma d_{\mu\sigma}(x)\, d_{\sigma\rho}(\psi) . \quad \square $$

I.1.11 Proposition: The right and the left action generators are related linearly through the adjoint representation

$$ {}^{R}X_\mu(x) = -\sum_\sigma {}^{t}d_{\mu\sigma}(x)\, {}^{L}X_\sigma(x) ; $$

$$ {}^{R}\Theta_{\mu\nu}(x) = \sum_{\sigma} {}^{t}d_{\mu\sigma}(x)\, {}^{L}\Theta_{\sigma\nu}(x) \,. $$

Proof: $$ {}^{R}\Theta_{\mu\nu}(x) = \frac{\partial \Phi_{\mu}^{e}}{\partial \xi_{\nu}}(\xi, x)\Big|_{\xi=0} $$

$$ = \frac{\partial \Phi_{\mu}^{e}}{\partial \nu_{\xi}}(x\,\xi\,x^{-1}, x)\Big|_{\xi=0} $$

$$ = \sum_{\delta} \frac{\partial \Phi_{\mu}^{e}}{\partial \delta_{\xi}}(\delta x \xi x^{-1}, x)\Big|_{\xi=0} \frac{\partial \delta_{\xi}^{e}}{\partial \xi_{\nu}}(\delta, x)\Big|_{\delta=0} $$

$$ = \sum_{\delta} {}^{t}d_{\delta\mu}(x)\, {}^{L}\Theta_{\delta\nu}(x) \qquad \text{(I.1.9), (I.1.2);} $$

$$ {}^{R}X_{\mu}(x) = \sum_{\nu} {}^{R}\Theta_{\mu\nu}(x) \frac{\partial}{\partial x_{\nu}^{e}} $$

$$ = -\sum_{\delta} {}^{t}d_{\delta\mu}(x) \sum_{\nu} \left[-{}^{L}\Theta_{\delta\nu}(x) \frac{\partial}{\partial x_{\nu}^{e}} \right] $$

$$ = -\sum_{\delta} {}^{t}d_{\delta\mu}(x)\, {}^{L}X_{\delta} \qquad \text{(I.1.3).} \quad \square $$

The next step is to determine explicitly matrices inverse to ${}^{R}\Theta$ and ${}^{L}\Theta$.

I.1.12 Definition: Let us define matrix $\Omega_{\delta\nu}(x)$

$$ \Omega_{\delta\nu}(x) := \frac{\partial}{\partial \xi_{\delta}} \Phi_{\nu}(x^{-1}, \tilde{\xi})\Big|_{\tilde{\xi}=x} \,. $$

I.1.13 Proposition: The matrix $\Omega(x)$ of definition I.1.12 is equal to the matrix ${}^{R}\Psi(x)$ already introduced in definition I.1.8,

$$ \Omega_{\delta\nu}(x) = {}^{R}\Theta^{-1}_{\delta\nu}(x) =: {}^{R}\Psi_{\delta\nu}(x) \,. $$

Proof:

$$ \frac{\partial}{\partial x_{\nu}} \Phi_{\sigma}(\alpha, x) \equiv \frac{\partial}{\partial \tau_{\nu}} \Phi_{\sigma}(\alpha, \tau)\Big|_{\tau=x} $$

$$ = \frac{\partial}{\partial \tau_{\nu}} \Phi_{\sigma}(\alpha, x x^{-1} \tau)\Big|_{\tau=x} $$

$$= \sum_\rho \frac{\partial}{\partial \tau_\nu} \Phi_\rho(x^{-1}, \tau)\big|_{\tau=x} \frac{\partial}{\partial \varepsilon_\rho} \Phi_\sigma(\alpha x, \varepsilon)\,\big|_{\varepsilon=0}$$

$$= \sum_\rho \Omega_{\nu\rho}(x)\, {}^R\Theta_{\rho\sigma}(\alpha x) \qquad \text{(I.1.12), (I.1.6).}$$

Put $\alpha = 0$ for the unit element of G, hence

$$\sum_\rho \Omega_{\nu\rho}(x)\, {}^R\Theta_{\rho\sigma}(x) = \delta_{\nu\sigma}\,.$$

$$\delta_{\nu\sigma} = \frac{\partial}{\partial \alpha_\nu} \Phi_\sigma(0, \alpha)$$

$$\equiv \frac{\partial}{\partial \alpha_\nu} \Phi_\sigma(0, \alpha)\big|_{\alpha=0}$$

$$= \frac{\partial}{\partial \alpha_\nu} \Phi_\sigma(x^{-1}x, \alpha)\big|_{\alpha=0}$$

$$= \sum_\rho \frac{\partial}{\partial \alpha_\nu} \Phi_\rho(x, \alpha)\big|_{\alpha=0} \frac{\partial}{\partial \tau_\rho} \Phi_\sigma(x^{-1}, \tau)\big|_{\tau=x}$$

$$= \sum_\rho {}^R\Theta_{\nu\rho}(x)\, \Omega_{\rho\sigma}(x) \qquad \text{(I.1.6), (I.1.12).}$$

□

I.1.14 Proposition: The matrix ${}^L\Psi(x)$ of definition I.1.4 is linearly related with the matrix ${}^R\Psi(x)$ through the adjoint representation

$${}^L\Psi(x) = {}^R\Psi(x)\, {}^t d(x).$$

Proof:

$$I = {}^R\Psi(x)\, {}^R\Theta(x) \qquad \text{(I.1.8)}$$

$$= {}^R\Psi(x)\, {}^t d(x)\, {}^L\Theta(x) \qquad \text{(I.1.11)}$$

$$I = {}^R\Theta(x)\, {}^R\Psi(x)$$

$$= {}^t d^{-1}(x)\, {}^R\Theta(x)\, {}^R\Psi(x)\, {}^t d(x)$$

$$= {}^L\Theta(x)\, {}^R\Psi(x)\, {}^t d(x) \qquad \text{(I.1.11).}$$

□

I.1.15 Proposition: Given the structure function $\Phi(\alpha,\beta)$, the matrix ${}^{L}\Psi(x)$ is explicitly determined as

$${}^{L}\Psi_{\nu\rho}(x) = \frac{\partial}{\partial\beta_\nu}\,\Phi_\rho(\beta, x^{-1})\Big|_{\beta=x}.$$

Proof:

$${}^{L}\Psi_{\nu\rho}(x) = \sum_\sigma {}^{R}\Psi_{\nu\sigma}(x)\,{}^{t}d_{\sigma\rho}(x) \qquad \text{(I.1.14)}$$

$$= \sum_\sigma \frac{\partial}{\partial\beta_\nu}\Phi_\sigma(x^{-1},\beta)\Big|_{\beta=x}\frac{\partial}{\partial\tau_\sigma}\Phi_\rho(x\tau,x^{-1})\Big|_{\tau=0} \quad \text{(I.1.12), (I.1.13), (I.1.9)}$$

$$= \frac{\partial}{\partial\beta_\nu}\,\Phi_\rho(\beta,x^{-1})\Big|_{\beta=x}. \quad \square$$

I.1.16 Proposition: If the group G that we consider is such that one can introduce exponential parametrization, then

$$\frac{\partial}{\partial\gamma_\beta}\,{}^{t}d_{\alpha\sigma}(\gamma)\Big|_{\gamma=0} = C^{\sigma}_{\beta\alpha} \quad , \text{ where } \quad d_{\alpha\sigma}(\gamma)$$

is the adjoint representation and $C^{\sigma}_{\beta\alpha}$ are structure constants. (See ref. 36 p. 102, 103 relations (3.14) and (3.23) ; the relation between $C^{\sigma}_{\beta\alpha}$ and ${}^{L}X_\alpha$ defined in (I.1.3) is: $[{}^{L}X_\alpha, {}^{L}X_\beta] = \sum_\rho C^{\rho}_{\alpha\beta}\,{}^{L}X_\rho$. From now on we are going to indicate the use of this relation (I.1.16'))

Proof:

$$\frac{\partial}{\partial\gamma_\beta}\,{}^{t}d_{\alpha\sigma}(\gamma) = \frac{\partial}{\partial\gamma_\beta}\,\frac{\partial\Phi_\sigma}{\partial x_\alpha}(\gamma x\gamma^{-1})\Big|_{x=0} \qquad \text{(I.1.9)}$$

$$= \frac{\partial}{\partial\gamma_\beta}\left\{\sum_\delta \frac{\partial\Phi_\rho}{\partial x_\alpha}(\gamma,x)\Big|_{x=0}\frac{\partial\Phi_\sigma}{\partial\sigma_\delta}(\sigma,\gamma^{-1})\Big|_{\sigma=\gamma}\right\}$$

$$= \sum_\delta\left\{\frac{\partial^2\Phi_\rho(\gamma,x)}{\partial\gamma_\beta\partial x_\alpha}\Big|_{x=0}\frac{\partial\Phi_\sigma(\sigma,\gamma^{-1})}{\partial\sigma_\delta}\Big|_{\sigma=\gamma} + \frac{\partial\Phi_\rho(\gamma,x)}{\partial x_\alpha}\Big|_{x=0}\frac{\partial^2\Phi_\sigma(\sigma,\gamma^{-1})}{\partial\sigma_\delta\partial\gamma_\beta}\Big|_{\sigma=\gamma}\right\}$$

$$= \sum_\delta\left\{\frac{\partial^2\Phi_\rho(\gamma,x)}{\partial\gamma_\beta\partial x_\alpha}\Big|_{x=0}{}^{L}\Psi_{\delta\sigma}(\gamma) + {}^{R}\Theta_{\alpha\delta}(\gamma)\,\frac{\partial^2\Phi_\sigma(\sigma,\gamma^{-1})}{\partial\sigma_\delta\partial\gamma_\beta}\Big|_{\sigma=\gamma}\right\}$$

(I.1.15), (I.1.6);

$$\frac{\partial}{\partial\gamma_\beta}\,{}^t d_{\alpha\sigma}(\gamma)\Big|_{\gamma=0} = \frac{\partial^2 \Phi_\sigma(\gamma,\chi)}{\partial\gamma_\beta\,\partial\chi_\alpha}\Big|_{\substack{\chi=0\\ \gamma=0}} + \frac{\partial^2 \Phi_\sigma(\sigma,\gamma^{-1})}{\partial\sigma_\alpha\,\partial\gamma_\beta}\Big|_{\sigma=\gamma=0} \qquad \text{(I.1.15), (I.1.6}$$

Under the condition that we work with exponential parametrization we can write $\gamma^{-1} = -\gamma$ also for finite γ, so

$$\frac{\partial}{\partial\gamma_\beta}\,{}^t d_{\alpha\sigma}(\gamma)\Big|_{\gamma=0} = \frac{\partial^2 \Phi_\sigma(\gamma,\chi)}{\partial\gamma_\beta\,\partial\chi_\alpha}\Big|_{\substack{\chi=0\\ \gamma=0}} - \frac{\partial^2 \Phi_\sigma(\gamma,\chi)}{\partial\gamma_\alpha\,\partial\chi_\beta}\Big|_{\substack{\gamma=0\\ \chi=0}}$$

$$= c^{\sigma}_{\beta\alpha} \quad \text{(see ref. 36, p. 103, (3.23))}. \qquad \square$$

I.1.17 Proposition: The left and right action generators commute among each other

$$\left[{}^L X_\mu\,,\,{}^R X_\nu\right] = 0 \quad \forall\,\mu,\nu.$$

Proof: For the proof we use a global argument - associativity:

$$\left(\tilde g_1^{-1} g\right)\tilde g_2 = \tilde g_1^{-1}\left(g\,\tilde g_2\right), \quad \tilde g_1, \tilde g_2 \in \tilde G \quad g \in G \quad (\tilde G = G). \qquad \square$$

I.1.18 Proposition: The right action generators fulfill the same commutation relations with the same structure constants as the left action generators

$$\left[{}^R X_\alpha\,,\,{}^R X_\beta\right] = \sum_\rho c^{\rho}_{\alpha\beta}\,{}^R X_\rho \quad \forall\,\alpha,\beta.$$

Proof:

$$\left[{}^R X_\alpha\,,\,{}^R X_\beta\right] = -\sum_\rho \left[{}^R X_\alpha\,,\,{}^t d_{\beta\rho}\,{}^L X_\rho\right] \qquad \text{(I.1.11)}$$

$$= -\sum_\rho \sum_\nu {}^R\Theta_{\alpha\nu}(\gamma)\left[\frac{\partial}{\partial\gamma_\nu}\,{}^t d_{\beta\rho}(\gamma)\right]{}^L X_\rho - \sum_\rho {}^t d_{\beta\rho}\left[{}^R X_\alpha\,,\,{}^L X_\rho\right]$$

$$= \sum_\rho \sum_\nu \sum_\mu {}^R\Theta_{\alpha\nu}(\gamma)\left[\frac{\partial}{\partial\gamma_\nu}\,{}^t d_{\beta\rho}(\gamma)\right]{}^t d^{-1}_{\rho\mu}(\gamma)\,{}^R X_\mu \qquad \text{(I.1.17), (I.1.}$$

On the left hand side we have the commutator $\left[{}^R X_\alpha\,,\,{}^R X_\beta\right]$ and on the right hand side linear combination of ${}^R X_\mu$'s with certain coefficients which depend on γ. From the fact that ${}^R X$'s are Lie-group generators we know that those coefficients should

be constant after the Lie's second theorem (see ref. 36 p. 100; the proof does not depend on whether the generators are left or right), so we take them on the place $\gamma = 0$, and obtain

$$\left[{}^R X_\alpha, {}^R X_\beta\right] = \sum_\mu \frac{\partial}{\partial \gamma_\alpha} {}^t d_{\beta\mu}(\gamma)\Big|_{\gamma=0} {}^R X_\mu$$

$$= \sum_\mu C_{\alpha\beta}^{\mu}\, {}^R X_\mu \qquad \text{(I.1.16)}. \qquad \square$$

Proposition I.1.18 can be proved alternatively, without the use of proposition I.1.17, but the proof is somewhat longer:

alternative proof: $\left[{}^R X_\alpha, {}^R X_\beta\right] = \sum_\nu \sum_\mu \left[{}^t d_{\alpha\nu}(\gamma)\, {}^L X_\nu, {}^t d_{\beta\mu}(\gamma)\, {}^L X_\mu\right]$ (I.1.11)

$$= \sum_\nu \sum_\mu \Big[-{}^t d_{\alpha\nu}(\gamma) \sum_\delta \Theta_{\nu\delta}(\gamma) \left(\frac{\partial}{\partial \gamma_\beta} {}^t d_{\delta\mu}(\gamma)\right) {}^L X_\mu + {}^t d_{\alpha\nu}(\gamma)\, {}^t d_{\beta\mu}(\gamma)\, {}^L X_\nu\, {}^L X_\mu$$

$$+ {}^t d_{\beta\mu}(\gamma) \sum_\delta \Theta_{\mu\delta}(\gamma) \left(\frac{\partial}{\partial \gamma_\beta} {}^t d_{\delta\nu}(\gamma)\right) {}^L X_\nu - {}^t d_{\beta\mu}(\gamma)\, {}^t d_{\alpha\nu}(\gamma)\, {}^L X_\mu\, {}^L X_\nu \Big] +$$

$$= \sum_\nu \sum_\mu \sum_\delta \Big[{}^t d_{\alpha\nu}(\gamma) \Theta_{\nu\delta}(\gamma) \left(\frac{\partial}{\partial \gamma_\beta} {}^t d_{\delta\mu}(\gamma)\right) \sum_\sigma {}^t d_{\mu\sigma}^{-1} {}^R X_\sigma$$

$$- {}^t d_{\beta\mu}(\gamma) \Theta_{\mu\delta}(\gamma) \left(\frac{\partial}{\partial \gamma_\beta} {}^t d_{\delta\nu}(\gamma)\right) \sum_\sigma {}^t d_{\nu\sigma}^{-1} {}^R X_\sigma \Big]$$

$$+ \sum_\nu \sum_\mu \sum_\delta \sum_\sigma {}^t d_{\alpha\nu}(\gamma)\, {}^t d_{\beta\mu}(\gamma)\, C_{\nu\mu}^{\delta}\, {}^L X_\delta \qquad \text{(I.1.11), (I.1.16)}$$

$$= \sum_\nu \sum_\mu \sum_\delta \sum_\sigma \Big\{ {}^t d_{\alpha\nu}(\gamma) \Theta_{\nu\delta}(\gamma) \left(\frac{\partial}{\partial \gamma_\beta} {}^t d_{\delta\mu}(\gamma)\right) {}^t d_{\mu\sigma}^{-1}(\gamma)$$

$$- {}^t d_{\beta\mu}(\gamma) \Theta_{\mu\delta}(\gamma) \left(\frac{\partial}{\partial \gamma_\beta} {}^t d_{\delta\nu}(\gamma)\right) {}^t d_{\nu\sigma}^{-1}(\gamma)$$

$$- {}^t d_{\alpha\nu}(\gamma)\, {}^t d_{\beta\mu}(\gamma)\, C_{\nu\mu}^{\delta}\, {}^t d_{\delta\sigma}^{-1}(\gamma) \Big\}\, {}^R X_\sigma \,.$$

Now we apply Lie's second theorem which tells us that coefficients on the right hand side in front of ${}^{R}X_{\sigma}$'s are constant, and we calculate those coefficients on the place $\gamma=0$ (see ref. 36 p. 100, and "comment" on p. 103).

$$\left[{}^{R}X_{\alpha}, {}^{R}X_{\beta}\right] = \sum_{\sigma} \left\{ \frac{\partial}{\partial\gamma_{\alpha}} {}^{t}d_{\beta\sigma}(\gamma)\Big|_{\gamma=0} - \frac{\partial}{\partial\gamma_{\beta}} {}^{t}d_{\alpha\sigma}(\gamma)\Big|_{\gamma=0} - C^{\sigma}_{\alpha\beta} \right\} {}^{R}X_{\sigma}$$

$$= \sum_{\sigma} C^{\sigma}_{\alpha\beta}\, {}^{R}X_{\sigma} \qquad \text{(I.1.16).} \qquad \square$$

Invariant volume element: Finally let us introduce the invariant volume element dm for the connected components of a continuous group, or shorter, for the Lie group.

I.1.19 Definition*: Invariant volume element (sometimes also called invariant measure) is defined

$$dm := \rho_{L,R}(\chi)\; dV_{L,R}(\chi),$$

where $dV_{L,R}(\chi) \equiv d_{L,R}\chi$ is obtained through left, right translation respectively of $dV(0)$ to the point χ of the group manifold; $\rho_{L,R}(\chi)$ is called density or Haar measure and it is defined through the same relation in a sense to provide us with the dm constant (invariant) over the group manifold ($dm = dV(0)$ for $\rho(0)=1$).

I.1.20 Proposition**: The density function (measure) for left, right translation (in the case that $\rho(0)=1$) is given by

$$\rho_{L}(\chi) = \det {}^{R}\Psi(\chi)\,, \quad \rho_{R}(\chi) = \det {}^{L}\Psi(\chi) \qquad \text{respectively.}$$

I.1.21 Proposition***: For the groups which have $\det d(\chi)=1$ ($d(\chi)$ being adjoint representation), the right measure is equal to the left measure

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* see ref. 36 p. 80-83

** see ref. 36 p. 81

*** see ref. 37 p. 366, ref. 38 p. 43, ref. 36 p. 81

$\rho_L = \rho_R$.

Proof: Recall (I.1.14),

$$L\Psi(\chi) = {}^R\Psi(\chi)\,{}^t d(\chi)\,. \quad \square$$

I.2. The action of a Lie group on its parameter space described by a subgroup and a coset space

If H is a non-trivial subgroup of G and C the right coset G/H, we may take G in the form $G = HC$ and get special forms for the matrices Θ and Ψ for right and left action of $\tilde{G}$ on G where $\tilde{G}$ equals G. The parameters of the group G are taken to be $\gamma_\mu, \gamma_{\mu'}, \ldots$ and determine $g(\gamma) \in G$; the parameters of the subgroup H are $\delta_\varkappa, \delta_{\varkappa'}, \ldots, \delta_\lambda, \delta_{\lambda'}, \ldots$ and determine $h(\delta) \in H$; the parameters of the right coset space $G/H \equiv C$ are $\beta_\tau, \beta_{\tau'}, \ldots, \beta_\sigma, \beta_{\sigma'}, \ldots$ and determine $c(\beta) \in C$. In what follows, the indices $\varkappa, \varkappa', \ldots, \lambda, \lambda', \ldots$ will always refer to the subgroup parameters δ, the indices $\tau, \tau', \ldots$ and $\sigma, \sigma', \ldots$ refer to the right coset parameters β, and the indices $\mu, \mu', \ldots$ refer to the parameters γ of the full group.
We start with the right action, $HC \times \tilde{G} \to HC$. The right action generators can be written

$${}^R X_\mu = \sum_\lambda {}^R\Theta_{\mu\lambda}(\gamma) \frac{\partial}{\partial \delta_\lambda} + \sum_\tau {}^R\Theta_{\mu\tau}(\gamma) \frac{\partial}{\partial \beta_\tau}\,.$$

It can be easily shown that ${}^R\Theta_{\mu\tau}(\gamma) \equiv {}^R\Theta_{\mu\tau}(\beta)$ and analogously ${}^R\Psi_{\sigma\mu}(\gamma) = {}^R\Psi_{\sigma\mu}(\beta)$. We are particularly interested in the right action of the group $\tilde{G}$ on some right coset space $C = G/H$. So we write the known proposition.

I.2.1 Proposition: The manifold M which supports the group action may be chosen to be the right coset space G/H under the right action of $\tilde{G}$ according to $C \times \tilde{G} \to C$. The right action generators are

$${}^R X_\mu(\beta) = \sum_\tau {}^R\Theta_{\mu\tau}(\beta) \frac{\partial}{\partial \beta_\tau}$$

and the corresponding ${}^{R}\Psi(\beta)$ and ${}^{R}\Theta(\beta)$ matrices are related by

$$\sum_{\mu} {}^{R}\Psi_{\sigma\mu}(\beta)\,{}^{R}\Theta_{\mu\tau}(\beta) = \delta_{\sigma\tau}.$$

One can understand what it means to reduce the right action of $\widetilde{G}$ on G to the right action of $\widetilde{G}$ on the right coset C from the global argument:

$g(\delta) = h(\delta)c(\beta)$, $g\tilde{g} = hc\tilde{g} = hh'c'$, which is equivalent to $c(\beta)\,\tilde{g} \to c'(\beta')$.

We consider now the left action $\widetilde{G} \times G \to G$.

I.2.2 Proposition: Within the new notation, the matrices ${}^{L}\Theta(\delta\beta)$ and ${}^{L}\Psi(\delta\beta)$ take the following block forms

$${}^{L}\Theta_{\varkappa\lambda}(\delta\beta) = {}^{L}\Theta_{\varkappa\lambda}(\delta), \qquad {}^{L}\Theta_{\varkappa\tau}(\delta\beta) = 0,$$

$${}^{L}\Theta_{\sigma\lambda}(\delta\beta) = \frac{\partial}{\partial\tilde{\beta}_\sigma}\Phi_\lambda(\tilde{\beta},\delta\beta)\Big|_{\tilde{\beta}=0}, \qquad {}^{L}\Theta_{\sigma\tau}(\delta\beta) = \frac{\partial}{\partial\tilde{\beta}_\sigma}\Phi_\tau(\tilde{\beta},\delta\beta)\Big|_{\tilde{\beta}=0};$$

$${}^{L}\Psi_{\varkappa\lambda}(\delta\beta) = {}^{L}\Psi_{\varkappa\lambda}(\delta), \qquad {}^{L}\Psi_{\varkappa\tau}(\delta\beta) = 0,$$

$${}^{L}\Psi_{\sigma\lambda}(\delta\beta) = \frac{\partial}{\partial\tilde{\beta}_\sigma}\Phi_\lambda(\delta\tilde{\beta},\beta^{-1}\delta^{-1})\Big|_{\tilde{\beta}=\beta}, \qquad {}^{L}\Psi_{\sigma\tau}(\delta\beta) = \frac{\partial}{\partial\tilde{\beta}_\sigma}\Phi_\tau(\tilde{\beta},\beta^{-1}\delta^{-1})\Big|_{\tilde{\beta}=\beta}$$

Proof:

$${}^{L}\Theta_{\varkappa\lambda}(\delta\beta) = \frac{\partial}{\partial\tilde{\delta}_\varkappa}\Phi_\lambda(\tilde{\delta}\tilde{\beta},\delta\beta)\Big|_{\substack{\tilde{\delta}=0\\ \tilde{\beta}=0}}$$

$$= \frac{\partial}{\partial\tilde{\delta}_\varkappa}\Phi_\lambda(\tilde{\delta},\delta)\Big|_{\tilde{\delta}=0} \equiv {}^{L}\Theta_{\varkappa\lambda}(\delta) \qquad \text{(I.1.2)},$$

$${}^{L}\Theta_{\varkappa\tau}(\delta\beta) = \frac{\partial}{\partial\tilde{\delta}_\varkappa}\Phi_\tau(\tilde{\beta},\delta\beta)\Big|_{\tilde{\beta}=0} = 0,$$

$${}^{L}\Theta_{\sigma\mu}(\delta\beta) = \frac{\partial}{\partial\tilde{\beta}_\sigma}\Phi_\mu(\tilde{\beta},\delta\beta)\Big|_{\tilde{\beta}=0};$$

$$ {}^{L}\Psi_{\varkappa\lambda}(\delta\beta) = \frac{\partial}{\partial\tilde{\delta}_{\varkappa}} \Phi_{\lambda}(\tilde{\delta}\tilde{\beta}, \beta^{-1}\delta^{-1}) \Big|_{\substack{\tilde{\delta}=\delta \\ \tilde{\beta}=\beta}} $$

$$ = \frac{\partial}{\partial\tilde{\delta}_{\varkappa}} \Phi_{\lambda}(\tilde{\delta}, \delta^{-1}) \Big| \tilde{\delta}=\delta \equiv {}^{L}\Psi_{\varkappa\lambda}(\delta) \qquad \text{(I.1.15)}, $$

${}^{L}\Psi_{\varkappa\tau}(\delta\beta)=0$ is clear from the fact that the matrices of a block form as ${}^{L}\Theta$ form the group,

$$ {}^{L}\Psi_{\sigma\lambda}(\delta\beta) = \frac{\partial}{\partial\tilde{\beta}_{\sigma}} \Phi_{\lambda}(\tilde{\delta}\tilde{\beta}, \beta^{-1}\delta^{-1}) \Big|_{\substack{\tilde{\delta}=\delta \\ \tilde{\beta}=\beta}}, $$

$$ {}^{L}\Psi_{\sigma\tau}(\delta\beta) = \frac{\partial}{\partial\tilde{\beta}_{\sigma}} \Phi_{\tau}(\tilde{\beta}, \beta^{-1}\delta^{-1}) \Big|_{\tilde{\beta}=\beta}. \quad \square $$

I.2.3 Proposition: The matrices ${}^{L}\Theta(\delta\beta)$ and ${}^{L}\Psi(\delta\beta)$ given in proposition I.2.2 are related in the following way to each other

$$ \sum_{\lambda'} {}^{L}\Psi_{\varkappa\lambda'}(\delta)\,{}^{L}\Theta_{\lambda'\lambda}(\delta) = \sum_{\lambda'} {}^{L}\Theta_{\varkappa\lambda'}(\delta)\,{}^{L}\Psi_{\lambda'\lambda}(\delta) = \delta_{\varkappa\lambda} $$

$$ \sum_{\tau'} {}^{L}\Psi_{\sigma\tau'}(\delta\beta)\,{}^{L}\Theta_{\tau'\tau}(\delta\beta) = \sum_{\tau'} {}^{L}\Theta_{\sigma\tau'}(\delta\beta)\,{}^{L}\Psi_{\tau'\tau}(\delta\beta) = \delta_{\sigma\tau} $$

$$ \sum_{\lambda} {}^{L}\Theta_{\sigma\lambda}(\delta\beta)\,{}^{L}\Psi_{\lambda\varkappa}(\delta) + \sum_{\tau} {}^{L}\Theta_{\sigma\tau}(\delta\beta)\,{}^{L}\Psi_{\tau\varkappa}(\delta\beta) = 0 $$

When one puts $\delta \equiv e(\sigma)$, the two last relations become

$$ \sum_{\tau'} {}^{L}\Psi_{\sigma\tau'}(\beta)\,{}^{L}\Theta_{\tau'\tau}(\beta) = \sum_{\tau'} {}^{L}\Theta_{\sigma\tau'}(\beta)\,{}^{L}\Psi_{\tau'\tau}(\beta) = \delta_{\sigma\tau}, $$

$$ {}^{L}\Theta_{\sigma\varkappa}(\beta) + \sum_{\tau} {}^{L}\Theta_{\sigma\tau}(\beta)\,{}^{L}\Psi_{\tau\varkappa}(\beta) = 0. $$

Proof: We use the definition I.1.4 which introduces ${}^{L}\Psi(\gamma)$ as ${}^{L}\Theta^{-1}(\gamma)$, and the explicit block form of ${}^{L}\Theta(\delta\beta)$ and ${}^{L}\Psi(\delta\beta)$ given in (I.2.2). $\square$

I.2.4 Proposition: The invariant volume element (invariant measure) on the right coset space G/H for the group having $\det d(\delta)=1$ has the form

$$dm(\beta) = \det {}^{L}\Psi_{\sigma\tau}(\beta)\, d\beta .$$

Proof: $dm(\delta\beta) = \rho(\delta\beta)\, d\beta\, d\delta$. Because of the property $\det d(\delta)=1$ we can use for $\rho(\delta\beta)$ both $\det {}^{L}\Psi(\delta\beta)$ and $\det {}^{R}\Psi(\delta\beta)$. We choose $\det {}^{L}\Psi(\delta\beta)$ and get

$$dm(\delta\beta) = \det {}^{L}\Psi(\delta\beta)\, d\beta\, d\delta$$

$$= \det {}^{L}\Psi_{\varkappa\lambda}(\delta)\, d\delta \det {}^{L}\Psi_{\sigma\tau}(\delta\beta)\, d\beta \qquad (I.2.2)$$

Putting $\delta = e(0)$, we get

$$dm(\beta) = \det {}^{L}\Psi_{\sigma\tau}(\beta)\, d\beta . \quad \square$$

Analogous results to those given in this section can be derived if $G = HC$ is replaced by $G = CH$ where C is a left coset space G/H.

We introduce the notation for the left coset space the parameters of the left coset space are ω_τ , $\omega_{\tau'}$, . . . ω_σ, $\omega_{\sigma'}$, ... and determine $c(\omega) \in C$. Propositions analgous to I.2.1, I.2.3 and I.2.4, which we enumerate I.2.1', I.2.3' and I.2.4' can be trivialy obtained writing everywhere ω instead of β and $\omega\delta$ instead of $\delta\beta$. One should also interchange indices "R" and "L" for right and left action. The proposition corresponding to I.2.2 is as follows:

I.2.2' Proposition: Within the new notation, the matrices ${}^{R}\Theta(\omega\delta)$ and ${}^{R}\Psi(\omega\delta)$ have the following properties:

$${}^{R}\Theta_{\varkappa\lambda}(\omega\delta) = {}^{R}\Theta_{\varkappa\lambda}(\delta) , \qquad {}^{R}\Theta_{\varkappa\tau}(\omega\delta) = 0 ,$$

$${}^{R}\Theta_{\sigma\lambda}(\omega\delta) = \frac{\partial}{\partial\tilde{\omega}_\sigma} \Phi_\lambda(\omega\delta,\tilde{\omega})\Big|_{\tilde{\omega}=0} , \qquad {}^{R}\Theta_{\sigma\tau}(\omega\delta) = \frac{\partial}{\partial\tilde{\omega}_\sigma} \Phi_\tau(\omega\delta,\tilde{\omega})\Big|_{\tilde{\omega}=0} ;$$

$${}^{R}\Psi_{\varkappa\lambda}(\omega\delta) = {}^{R}\Psi_{\varkappa\lambda}(\delta) , \qquad {}^{R}\Psi_{\varkappa\tau}(\omega\delta) = 0 ,$$

$${}^{R}\Psi_{\sigma\lambda}(\omega\delta) = \frac{\partial}{\partial\tilde{\omega}_\sigma} \Phi_\lambda(\delta^{-1}\omega^{-1},\tilde{\omega}\delta)\Big|_{\tilde{\omega}=\omega} , \qquad {}^{R}\Psi_{\sigma\tau}(\omega\delta) = \frac{\partial}{\partial\tilde{\omega}_\sigma} \Phi_\tau(\delta^{-1}\omega^{-1},\tilde{\omega})\Big|_{\tilde{\omega}=\omega} .$$

I.3. Parameter space for the matrix unitary group

So far, a general group G was dealt with. Now we switch to the unitary group U(j), which is exponentially parametrized. The well-known* setting with complex parameters for the matrix group U(j) is given $\Psi : \Psi_{ie} = -\overline{\Psi_{ei}}$. The corresponding matrix generators are $\{\mathcal{C}_{ie}\}: (\mathcal{C}_{ie})^{+} = \mathcal{C}_{ei}$ where + means hermitian conjugate, and they fulfil the following commutation relations: $[\mathcal{C}_{ie}, \mathcal{C}_{jk}] = \mathcal{C}_{ik}\delta_{ej} - \mathcal{C}_{je}\delta_{ik}$ and the generators are given as matrices
Any U(j) generator can be written: $X = \sum_{i,e} \Psi_{ie} \mathcal{C}_{ie}$.
Let us determine the transition to the setting with real parameters.

I.3.1 Definition:

$$\Psi_{i<e} (\equiv \Psi_{<}) = (\varphi_r)_{(ie)} + i(\varphi_i)_{(ie)}$$

where $(\varphi_r)_{(ie)} := \mathrm{Re}(\Psi_{i<e})$ and $(\varphi_i)_{(ie)} := \mathrm{Im}(\Psi_{i<e})$**

$$\Psi_{e>i} (\equiv \Psi_{>}) = -(\varphi_r)_{(ie)} + i(\varphi_i)_{(ie)} ;$$

$$\Psi_{ii} (\equiv \Psi_{o}) =: i(\varphi_{\bullet})_{(ii)} , \quad \mathrm{Re}(\Psi_{ii}) \equiv 0.$$

Similar relations can be written in terms of the structure functions

$$\Phi_{i<e} (\equiv \Phi_{<}(\Psi', \Psi'')) = (\Phi_r)_{(ie)} + (\Phi_i)_{(ie)}$$

where $(\Phi_r)_{(ie)} := \mathrm{Re}(\Phi_{i<e})$ and $(\Phi_i)_{(ie)} := \mathrm{Im}(\Phi_{i<e})$;

$$\Phi_{e>i} (\equiv \Phi_{>}(\Psi', \Psi'')) = -(\Phi_r)_{(ie)} + i(\Phi_i)_{(ie)}$$

$$\Phi_{ii} (\equiv \Phi_{o}(\Psi', \Psi'')) =: i(\Phi_{\bullet})_{(ii)} , \quad \mathrm{Re}(\Phi_{ii}) \equiv 0.$$

---

*See, for example ref. 36.

**Two latin indices in brackets now play the role of one greek index, $(ie) \leftrightarrow \mu$, for real group parameters.

I.3.2 Proposition: The matrix P which provides us with the transition from the real to the complex indices is given by

$$(\psi_o \psi_< \psi_>) = (\varphi_o \varphi_r \varphi_i) P$$
$$(\Phi_o \Phi_< \Phi_>) = (\Phi_o \Phi_r \Phi_i) P$$

$$P_{\mu;il} \equiv P_{(nk),il} \rightarrow \begin{matrix} \\ (nn) \\ r(nk) \\ i(nk) \end{matrix} \begin{matrix} \begin{matrix} nn & nk & kn \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & i & i \end{bmatrix} \end{matrix}, \quad n<k .$$

I.3.3 Proposition: The corresponding derivatives are related by

$$\begin{bmatrix} \frac{\partial}{\partial \psi_o} \\ \frac{\partial}{\partial \psi_<} \\ \frac{\partial}{\partial \psi_>} \end{bmatrix} = \begin{bmatrix} -i & 0 & 0 \\ 0 & 1/2 & -i/2 \\ 0 & -1/2 & -i/2 \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial \varphi_o} \\ \frac{\partial}{\partial \varphi_r} \\ \frac{\partial}{\partial \varphi_i} \end{bmatrix} = P^{-1} \begin{bmatrix} \frac{\partial}{\partial \varphi_o} \\ \frac{\partial}{\partial \varphi_r} \\ \frac{\partial}{\partial \varphi_i} \end{bmatrix}$$

I.3.4 Definition: We define matrix generators in the real setting:

$L_{(i<e)} := c_{ie} - c_{ei} = c_{ie} - (c_{ie})^+$, which generate the subgroup SO(j)

$D_{(i<e)} := i\,(c_{ie} + (c_{ie})^+)$

$D_{(ii)} := i\, c_{ii}$

This relation can be written as

$$\begin{bmatrix} D_{(ii)} \\ L_{(i<e)} \\ D_{(i<e)} \end{bmatrix} = P \begin{bmatrix} c_{ii} \\ c_{ie} \\ (c_{ie})^+ \end{bmatrix},$$

and any U(j) matrix generator can be written

$$X = \sum_{(i<e)} \sum_{l} \left[(\varphi_r)_{(ie)} L_{(ie)} + (\varphi_i)_{(ie)} D_{(ie)} \right] + \sum_i (\varphi_o)_{(ii)} D_{(ii)} .$$

Clearly the same relations holds true also for differential infinitesimale generators of the U(j).

$$\Theta_{ie,\mu}(\varphi) = \frac{\partial}{\partial \tilde{\varphi}_{ie}} \Phi_\mu \Big|_{\tilde{\varphi}=0}$$

$$= -\frac{\partial}{\partial \tilde{\varphi}_{ei}} \Phi_\mu \Big|_{\tilde{\varphi}=0}$$

$$= -\overline{\left(\frac{\partial}{\partial \tilde{\varphi}_{ei}} \Phi_\mu \right)}\Big|_{\varphi=0}$$

$$= -\overline{\Theta_{ei,\mu}(\varphi)}. \quad \square$$

The matrices ${}^L\Psi(\varphi)$, ${}^R\Psi(\varphi)$ and $d(\varphi)$ were up to now expressed through certain derivatives of the structure function. The next propositions will provide us with their expressions through the matrix elements of the fundamental representation [1] of U(j).

I.3.12 Proposition:

$${}^L\Psi_{\mu,ie}(\varphi) = \sum_{k=1}^{\dot{j}} \left[\frac{\partial}{\partial \varphi_\mu} u_{ik}(\varphi) \right] u^+_{ke}(\varphi) \quad , \quad u(\varphi) \in U(j) \, , \; \varphi = \varphi(\varphi).$$

Proof:

$$\sum_k \left[\frac{\partial}{\partial \varphi_\mu} u_{ik}(\varphi) \right] u^+_{ke}(\varphi) = \frac{\partial}{\partial \tilde{\varphi}_\mu} \left[\sum_k u_{ik}(\hat{\Phi}(\hat{\varphi})) \, u_{ke}(\varphi^{-1}) \right] \Big|_{\tilde{\varphi}=\varphi}$$

$$= \left[\frac{\partial}{\partial \tilde{\varphi}_\mu} u_{ie}\left(\Phi(\hat{\varphi}, \varphi^{-1}) \right) \right]_{\tilde{\varphi}=\varphi}$$

$$= \frac{\partial}{\partial \tilde{\varphi}_\mu} \left[Exp \sum_{u,v=1}^{\dot{g}} \Phi_{uv}(\hat{\varphi}, \varphi^{-1}) \, c_{uv} \right]_{ie} \Big|_{\tilde{\varphi}=\varphi}$$

Now we use

$$\left[Exp \sum_{u,v} \Phi_{uv}(\tilde{\varphi}, \varphi^{-1}) c_{uv} \right]_{ie} = \delta_{ie} + \Phi_{ie}(\tilde{\varphi}, \varphi^{-1}) + \frac{1}{2} \sum_k \Phi_{ik} \, \Phi_{ke} + \dots$$

where the higher terms are not neglected, and one gets

$$\sum_k \left[\frac{\partial}{\partial \varphi_\mu} u_{ik}(\varphi) \right] u^+_{ke}(\varphi) = \frac{\partial}{\partial \tilde{\varphi}_\mu} \Phi_{ie}(\tilde{\varphi}, \varphi^{-1}) \Big|_{\tilde{\varphi}=\varphi}$$

$$= {}^L\Psi_{\mu,ie}(\varphi) \qquad \text{(I.1.15)}. \quad \square$$

I.3.13 Proposition: The adjoint representation can be written

$${}^t d_{km,ie}(\Psi(\varphi)) = u_{ik}(\Psi)\, u^+_{me}(\Psi).$$

Proof:

$$\frac{\partial}{\partial\beta_{km}} \Big[\mathrm{Exp}\Big(\sum_{u,v=1}^{j} \Phi_{uv}(\Psi\beta\Psi^{-1}) c_{uv}\Big)\Big]_{ie}\Big|_{\beta=0} = \frac{\partial}{\partial\beta_{km}} \Phi_{ie}(\Psi\beta\Psi^{-1})\Big|_{\beta=0}$$

$$= \sum_{\rho=1}^{j^2} \sum_{\mu=1}^{j^2} P^{-1}_{km,\rho} \frac{\partial}{\partial\alpha_\rho} \Phi_\mu(\Psi\beta\Psi^{-1})\Big|_{\beta=0} P_{\mu,ie} \qquad \text{(I.3.2), (I.3.3)}$$

$$= \sum_{\rho} \sum_{\mu} P^{-1}_{km,\rho}\, {}^t d_{\rho\mu}(\Psi)\, P_{\mu,ie} \qquad \text{(I.1.9)}$$

$$=: {}^t d_{km,ie}(\Psi).$$

Now one starts again with the first expression

$$\frac{\partial}{\partial\beta_{km}} \Big[\mathrm{Exp}\Big(\sum_{u,v} \Phi_{uv}(\Psi\beta\Psi^{-1}) c_{uv}\Big)\Big]\Big|_{\beta=0} =$$

$$= \sum_{t,n} u_{it}(\Psi) \Big[\Big(\frac{\partial}{\partial\beta_{km}} \big[\mathrm{Exp}\textstyle\sum \beta c\big]_{tn}\Big)\Big|_{\beta=0}\Big] u^+_{ne}(\Psi)$$

$$= \sum_{t,n} u_{it}(\Psi) \frac{\partial}{\partial\beta_{km}} \beta_{tn}\Big|_{\beta=0} u^+_{ne}(\Psi)$$

$$= u_{ik}\, u^+_{me}.$$

Hence ${}^t d_{km,ie}(\Psi(\varphi)) = u_{ik}(\Psi)\, u^+_{me}(\Psi)$. □

I.3.14 Proposition: ${}^t d_{km,ie} = {}^t\overline{d_{mk,ei}}$.

Proof: Recall (I.1.9) and the fact that $\Psi_{ie} = -\overline{\Psi_{ei}}$. □

I.3.15 Proposition: $\frac{\partial}{\partial\varphi_\mu} u_{ie}(\Psi) = \sum_{k=1}^{j} u_{ik}(\Psi)\, {}^R\Psi_{\mu,ke}(\Psi)$.

Proof:

$${}^R\Psi_{\mu,ie}(\Psi) = \sum_{k,m=1}^{j} {}^L\Psi_{\mu,km}(\Psi)\, {}^t d^{-1}_{km,ie}(\Psi) \qquad \text{(I.1.14), (I.3.2), (I.}$$

$$= \sum_{k=1}^{j} u_{ik}(\Psi^{-1}) \frac{\partial}{\partial\varphi_\mu} u_{ke}(\Psi) \qquad \text{(I.3.12), (I.3.13).}$$ □

I.4. The parameter space for the matrix SO(j) groups

Now we introduce SO(j) matrix group, that is, (100....)* the fundamental representation of SO(j) which is also exponentially parametrized. One of course uses the fact that $SO(j) < U(j)$ and chooses for the SO(j) the real parameters $(\varphi_r)_{(i<\ell)}$ defined in (I.3.1). The infinitesimale generators basis consists of $L_{(i<\ell)} := \kappa_{i\ell} - \kappa_{\ell i}$ for $i,1 = 1,2,..,j$ (see (I.3.4)), and any element of the Lie algebra can be written $X = \sum_{(i<\ell)} (\varphi_r)_{(i<\ell)} L_{(i<\ell)}$. It is also clear that the corresponding P matrix is trivialy an identity $((\varphi_r)_{(i<\ell)} \leftrightarrow \varphi_\mu)$. In the SO(j) case we are going to write $\Psi_{i<\ell}$ instead of $(\varphi_r)_{(i<\ell)}$ for simplicity, and in those places where we are not much interested in what is going on with a certain pair of indices, we write Ψ_μ instead of $\Psi_{i<\ell}$.

I.4.1 Proposition: $\sum_{k=1}^{j} \left[\frac{\partial}{\partial\psi_\mu} \sigma_{ik}(\psi)\right] {}^t\sigma_{k\ell}(\psi) =: {}^L\tilde{\Psi}_{\mu,i\ell}(\psi)$

$$= \begin{cases} {}^L\Psi_{\mu,i<\ell} & \text{for } i<\ell \\ 0 & \text{for } i=\ell \\ -{}^L\Psi_{\mu,\ell<i} & \text{for } i>\ell \end{cases} .$$

Proof: In an analogous way, as in the proof of proposition I.3.12, one shows that

$$\sum_{k=1}^{j} \left[\frac{\partial}{\partial\psi_\mu} \sigma_{ik}(\psi)\right] {}^t\sigma_{k\ell}(\psi) = \frac{\partial}{\partial\tilde{\psi}_\mu} \left[Exp \sum_{u=1}^{j} \sum_{v=u+1}^{j} \Phi_{uv}(\tilde{\psi},\psi^{-1}) (\kappa_{uv} - \kappa_{vu})\right]_{i\ell} \Big|_{\tilde{\psi}=\psi}$$

$$= \begin{cases} \frac{\partial}{\partial\tilde{\psi}_\mu} \Phi_{i\ell}(\tilde{\psi},\psi^{-1})\big|_{\tilde{\psi}=\psi} & \text{for } i<\ell \\ 0 & \text{for } i=\ell \\ -\frac{\partial}{\partial\tilde{\psi}_\mu} \Phi_{\ell i}(\tilde{\psi},\psi^{-1})\big|_{\tilde{\psi}=\psi} & \text{for } i>\ell \end{cases}$$

*See ref. 39 p. 396, 397

$$= \begin{cases} {}^{L}\mathcal{F}_{\mu,ie}(\Psi) & \text{for } i<e \\ 0 & \text{for } i=e \\ -{}^{L}\mathcal{F}_{\mu,ei}(\Psi) & \text{for } i>e \end{cases} \qquad \text{(I.1.15).}$$

I.4.2 Proposition: The adjoint representation ${}^{t}d_{<,<}$ can be written

$$\sigma_{ik}(\Psi){}^{t}\sigma_{me}(\Psi) - \sigma_{im}(\Psi){}^{t}\sigma_{ke}(\Psi) = \begin{cases} {}^{t}d_{km,ie} & \text{for } k<m,\ i<e \\ -{}^{t}d_{km,ei} & \text{for } k<m,\ i>e \\ -{}^{t}d_{mk,ie} & \text{for } k>m,\ i<e \\ {}^{t}d_{mk,ei} & \text{for } m<k,\ e<i \\ 0 & \text{for all other cases.} \end{cases}$$

Proof: The proof is analogous to the proof of (I.3.13). One shows that

$$\frac{\partial}{\partial\beta_{k<m}}\left[Exp\left\{\sum_{\substack{u,v \\ (u<v)}} \Phi_{u<v}(\Psi\beta\Psi^{-1})(\mathcal{C}_{uv}-\mathcal{C}_{vu})\right\}\right]_{i<e}\Big|_{\beta=0} =$$

$$= \frac{\partial}{\partial\beta_{k<m}}\Phi_{i<e}(\Psi\beta\Psi^{-1})\Big|_{\beta=0}$$

$$= {}^{t}d_{k<m,i<e} \qquad \text{(I.1.9).}$$

One starts again with the first expression and shows

$$\frac{\partial}{\partial\beta_{k<m}}\left\{Exp\left[\sum_{\substack{u,v \\ (u<v)}} \Phi_{u<v}(\Psi\beta\Psi^{-1})(\mathcal{C}_{uv}-\mathcal{C}_{vu})\right]\right\}_{i<e}\Big|_{\beta=0} =$$

$$= \sum_{t,n} \sigma_{it}(\Psi)\left\{\frac{\partial}{\partial\beta_{k<m}}\left[Exp\sum_{\substack{u,v \\ (u<v)}} \beta_{u<v}(\mathcal{C}_{uv}-\mathcal{C}_{vu})\right]_{tn}\right\}\Big|_{\beta=0}{}^{t}\sigma_{ne}(\Psi)\Big|_{i<e}$$

$$= \sum_{\substack{t,n \\ (t<n)}} \sigma_{it}(\psi)\, \frac{\partial \beta_{t<n}}{\partial \beta_{k<m}}\, {}^{t}\sigma_{ne} - \sum_{\substack{t,n \\ (t>n)}} \sigma_{it}\, \frac{\partial \beta_{n<t}}{\partial \beta_{k<m}}\, {}^{t}\sigma_{ne} \;\Big|_{i<\ell}$$

$$= \sigma_{ik}\, {}^{t}\sigma_{me} - \sigma_{im}\, {}^{t}\sigma_{ke} \;\Big|_{k<m,\, i<\ell}. \quad \square$$

I.4.3 Proposition: ${}^{R}\mathcal{F}_{\mu, i<\ell}(\psi) = \sum_{k=1}^{\dim} {}^{t}\sigma_{ik}(\psi)\, \frac{\partial}{\partial \psi_\mu}\, \sigma_{k\ell}(\psi)$.

Proof:

$${}^{R}\mathcal{F}_{\mu, i<\ell} = \sum_{\substack{k,m \\ (k<m)}} {}^{L}\mathcal{F}_{\mu, km}\, {}^{t}d^{-1}_{km,\, i<\ell} \qquad \text{(I.1.14)}$$

$$= \sum_{\substack{k,m \\ (k<m)}} \sum_{j} \left(\frac{\partial}{\partial \psi_\mu}\, \sigma_{kj}\right) {}^{t}\sigma_{jm} \left({}^{t}\sigma_{ik}\, \sigma_{me} - {}^{t}\sigma_{im}\, \sigma_{ke}\right)\Big|_{i<\ell} \qquad \text{(I.4.1), (I.4.}$$

$$= \sum_{k} \left(\frac{\partial}{\partial \psi_\mu}\, \sigma_{ke}\right) {}^{t}\sigma_{ik} \;\Big|_{i<\ell} \qquad \text{(I.4.1).}$$

I.4.4 Proposition: $\sum_{k=1}^{\dim} {}^{t}\sigma_{ik}\, \frac{\partial}{\partial \psi_\mu}\, \sigma_{k\ell} =: {}^{R}\tilde{\mathcal{F}}_{\mu, i\ell}$

$$= \begin{cases} {}^{R}\mathcal{F}_{\mu, i\ell} & \text{for } i<\ell \\ 0 & \text{for } i=\ell \\ -{}^{R}\mathcal{F}_{\mu, \ell i} & \text{for } i>\ell \end{cases}.$$

Proof: The relation for $i<\ell$ is already proved in (I.4.3). Proof for (I.4.4) is analagous to the one for (I.4.1). $\square$

I.4.5 Proposition: $\tilde{\mathcal{F}}_{\mu, i\ell} = -\tilde{\mathcal{F}}_{\mu, \ell i}$ both for the left and right action.

Proof: Follows immediately from (I.4.1) and (I.4.4). $\square$

I.4.6 Definition:

$$\tilde{\Theta}_{il,\mu} := \begin{cases} \Theta_{il,\mu} & \text{for } i<l \\ 0 & \text{for } i=l \\ -\Theta_{li,\mu} & \text{for } i>l \end{cases}$$

both for the left and right action.

I.4.7 Proposition: $\tilde{\Theta}_{il,\mu} = -\tilde{\Theta}_{li,\mu}$ both for the left and right action.

Proof: Follows from (I.4.6). □

II. The $SO(3)\times SO(n)$ scheme

II.0 Introduction

In this chapter we use the results of chapter I (and specially of (I.4)) to introduce the coordinate transformation based on the acting group $\widetilde{G}$ which is the direct product of the collective group $SO(3)$ and intrinsic group SO(n). This coordinate transformation can also be obtained from the $gl_+(3,\mathbb{R})\times SO(n)$ action as we are going to show in chapter III (section III.1). The result of this coordinate transformation is that the point in configuration space in new coordinates can be written uniquely decomposed as $\xi_{is} = \sum_{\ell=1}^{3} \sigma_{i\ell}(\omega)\,\mathring{\xi}_\ell\,\sigma'_{\ell s}(\beta)$ where $i,\ell = 1,2,3$ $s = 1,2,\ldots,n$ $(n \equiv A-1)$; $\sigma(\omega)\in SO(3)$; $\sigma'(\beta)\in SO(n)/SO(n-3)$, right coset space and $\mathring{\xi}_1 > \mathring{\xi}_2 > \mathring{\xi}_3 > 0$, real numbers. In this scheme ω's are 3 rotational parameters and $\mathring{\xi}_\ell$ are parameters for vibrations and deformations. Parameters β are the maximal set of coordinates which are left unchanged under the SO(n) right action. It becomes clear that this scheme is particularly convenient for Vanagas' collectivity, the term defined in the general introduction. According to the scheme, ω's and $\mathring{\xi}$'s span the transversal w.r. t. the SO(n) action and should be considered as collective coordinates. This scheme has been studied by many authors.[21,22,23,24] Some of them[23,24] tried in the frame of $SO(3)\times SO(n)$ scheme to introduce $gl_+(3,\mathbb{R})$ collectivity "adding" to 6 collective coordinates defined above three parameters which span the manifold SO(3), submanifold of the right coset space SO(n)/SO(n-3). They left the intrinsic coordinates not clearly defined, they stayed in the $SO(3)\times SO(n)$ scheme. We reconstruct their results using our method and then discuss in particular the possiblity of the complete separation of kinetic energy and velocity field in the collective and intrinsic parts w.r. t. Vanagas' and $gl_+(3,\mathbb{R})$ collectivity.

II.1. The $SO(3)\times SO(n)$ scheme: collective and intrinsic coordinates, orbits

Let us consider the system of n+1 particles moving in 3-dimensional Euclidean space E^3. The configuration of n+1 particles can be seen as the point in the 3(n+1)-dimensional Euclidean space $E^{3(n+1)}$. Considering the center-of-mass already separated, our problem is reduced to E^{3n}. We map E^{3n} on $\mathbb{R}^{3n}$ with a Cartesian coordinate chart, where each point of E^{3n} is represented by ξ_{is}, i = 1,2,3 is the space index and s = 1,2,...n the relative vector index.

We introduce the "collective" tensor

$$\mathcal{T}_{ij}(\xi) := \sum_{s=1}^{n} \xi_{is}\,\xi_{js}$$

which is an 3x3 real symmetric positive semidefinite matrix. Hence, there exists orthogonal 3x3 matrix $\sigma \in SO(3)$ which diagonalizes the "collective" tensor

$$\mathcal{T}(\xi) = \sigma\,\mu\,{}^t\sigma\,,$$

$$\mu_{ik} = \mu_i\,\delta_{ik} \qquad i,k = 1,2,3$$

and μ_1, μ_2 and μ_3 are nonnegative real numbers.

We define

$$\overset{\circ}{\xi}_{is} := \mu_i^{-\frac{1}{2}}\,\delta_{is} \quad i=1,2,3 \quad s=1,2,\ldots,n \text{ and}$$

$$\mu^{\frac{1}{2}}_{ij} := \mu_i^{\frac{1}{2}}\,\delta_{ij} \quad i,j=1,2,3$$

which are a 3xn real matrix and a 3x3 real matrix, respectively. Then

$$\mathcal{T}_{ij}(\overset{\circ}{\xi}) = \mu_i\,\delta_{ij}$$

Also using the expression

$$\mathcal{T}(\xi) = \sigma\,\mu\,{}^t\sigma$$

one concludes

$$\mathcal{T}(\xi) = \sigma\;\mathcal{T}(\overset{\circ}{\xi})\,{}^t\sigma$$

Let us define a 3xn real matrix $'\sigma'$

$$\mu^{\frac{1}{2}}\,{}^1\sigma' := {}^t\sigma\,\xi$$

and demand that $\mu_i > 0$ for i = 1,2,3, then

$${}^1\sigma'\,{}^t({}^1\sigma') = \mu^{-\frac{1}{2}}\,{}^t\sigma\,\xi\,{}^t\xi\,\sigma\,\mu^{-\frac{1}{2}} = \mu^{-\frac{1}{2}}\mu\mu^{-\frac{1}{2}} = \mathbb{I}_3 .$$

Later in this section we are going to discuss cases in which the μ_i's also take the value zero.
Now ${}^1\sigma'$ can be supplemented by (n-3) rows ${}^2\sigma'$ to yield an nxn matrix σ' satisfying

$$\sigma'\,{}^t\sigma' = \mathbb{I}_n$$

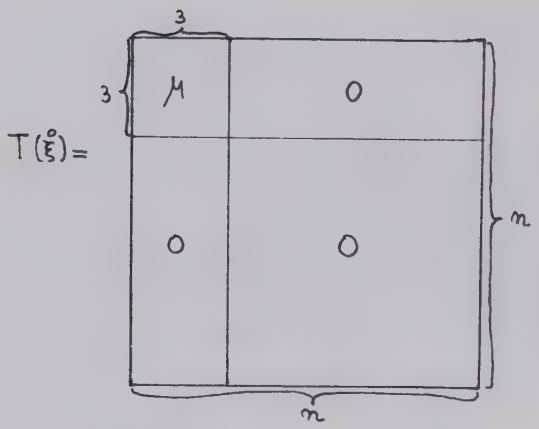
which means $\sigma' \in SO(n)$
With this matrix we can write

$$\xi = \sigma\,\overset{\circ}{\xi}\,\sigma'$$

where all symbols have already been defined.
All this has been done in order to obtain the new set of coordinates such that a part appears as some group parameters. But on the left-hand side of the last expression there are 3n Cartesian coordinates and on the right-hand side 3 SO(3) parameters plus 3 $\overset{\circ}{\xi}_i$'s plus $\frac{n(n-1)}{2}$ SO(n) parameters, which yields $\frac{n(n-1)}{2} + 6 > 3n$ parameters. In order to find exactly 3n new coordinates we start now with an orbit analysis. We proceed somewhat indirectly.
Let us introduce the "intrinsic" tensor which is an nxn real symmetric matrix

$$T(\xi) := {}^t\xi\,\xi$$

$$= {}^t\sigma'\,T(\overset{\circ}{\xi})\,\sigma' , \qquad \text{where}$$



Being equipped with two tensors $\mathcal{T}(\xi)$ and $T(\xi)$ we define the action of $SO(3) \times SO(n)$ on both of them choosing $\mathcal{T}(\mathring{\xi})$ and $T(\mathring{\xi})$ to be the representative points, respectively. We define the action of $SO(3) \times SO(n)$. The groups act on $\mathcal{T}(\xi)$ and $T(\xi)$ through the action defined on ξ :

$$\left(SO(3) \times SO(n)\right) \times \mathbb{R}^{3n} \to \mathbb{R}^{3n}$$

$$\left((\tilde{\sigma}, \tilde{\sigma}'), \xi\right) \to {}^{t}\tilde{\sigma}\, \xi\, \tilde{\sigma}' = \xi'$$

where $\tilde{\sigma} \in SO(3)$ and $\tilde{\sigma}' \in SO(n)$.
An action on $\mathcal{T}(\xi)$ is

$\left((\tilde{\sigma}, \tilde{\sigma}'), \mathcal{T}(\xi)\right) \to \mathcal{T}(\xi')$ is then

$$\mathcal{T}(\xi') := \mathcal{T}({}^{t}\tilde{\sigma}\, \xi\, \tilde{\sigma}')$$

$$= {}^{t}\tilde{\sigma}\; \mathcal{T}(\xi)\; \tilde{\sigma}$$

and reduces to the action of $SO(3)$ on $\mathcal{T}(\xi)$.

In an analogous way, one can see that the $SO(3) \times SO(n)$ action

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*Our orbit analysis uses two tensors $\mathcal{T}(\xi)$ and $T(\xi)$ and is in that sense similar to the analysis in reference 23 , although the authors in 23 don't explicitely do an orbit analysis.

on $T(\xi)$ reduces to an SO(n) action on $T(\xi)$

$T(\xi') = {}^t\tilde{\sigma}^{-1}\, T(\xi)\, {}^t\tilde{\sigma}^{-1}$.

One could now look for the stability group of the representative point $\mathcal{T}(\mathring{\xi})$ under the action of $SO(3)$. We call that group $\mathcal{H}_3'$. Analogously the SO(n) action on $T(\mathring{\xi})$ provides us with the stability group H_n'. If we now choose the representative point in $\mathbb{R}^{3n}$ under the action of $SO(3)\times SO(n)$, $\mathring{\xi}$, and call the stability group H, we conclude that

$$H < H' \qquad \text{where} \quad H' := \mathcal{H}_3' \times H_n'$$

So, for each type of the representative point $\mathring{\xi}$ we could look for the corresponding H', and then determine H.

We claim that only one type of representative point $\mathring{\xi}$ should be considered and we prove it. Let us first give the result for the orbit analysis with the representative point $\mathring{\xi}$ of the type $\mu_1^{\frac{1}{2}} > \mu_2^{\frac{1}{2}} > \mu_3^{\frac{1}{2}} > 0$. The stability group of $\mathcal{T}(\mathring{\xi})$ is $\mathcal{H}_3'$

$$\mathcal{H}_3' = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&-1&0\\0&0&-1 \end{pmatrix} \right\}$$

$\mathcal{H}_3' < SO(3)$; the stability group of $T(\mathring{\xi})$ is $H_n' = \mathcal{H}_3' \times SO(n-3)$, $H_n' < SO(n)$. It follows that $H = \left[\mathrm{diag}(\mathcal{H}_3' \times \mathcal{H}_3')\right] \times SO(n-3)$. The orbit obtained is (3n-3)-dimensional. Now we look at the transversal to the $SO(3)\times SO(n)$ action. The transversal is the submanifold left invariant under the group action. The transversal in our case is spanned by the $\mathring{\xi}$ coordinates. We show that $(\mu_1^{\frac{1}{2}}\ \mu_2^{\frac{1}{2}}\ \mu_3^{\frac{1}{2}})$ or equivalently $(\mathring{\xi}_1\ \mathring{\xi}_2\ \mathring{\xi}_3)$ is an ordered set. Suppose that both $(\mathring{\xi}_1\ \mathring{\xi}_2\ \mathring{\xi}_3)$ and the permuted point $(\mathring{\xi}_2\ \mathring{\xi}_1\ \mathring{\xi}_3)$ lie on the transversal. One sees immediately that the second point can be obtained through the action of

$$\begin{pmatrix} 0&1&0\\-1&0&0\\0&0&1 \end{pmatrix} \times \left(\begin{array}{ccc|c} 0&-1&0&\\1&0&0&0\\0&0&1& \\ \hline & 0 & & SO(n-3) \end{array}\right) \in SO(3)\times SO(n)$$

on the first point!

Now let us keep the representativ point where still all components $\mu_i^{\frac{1}{2}} > 0$ but where two values are equal. Then $\mathcal{H}_3' = \{(\sigma_2, \sigma_1) | \sigma_2 \in SO(2), \sigma_1 = \pm 1\}$, $H_n' = \mathcal{H}_3' \times SO(n-3)$ and finally $H = [diag(\mathcal{H}_3' \times \mathcal{H}_3')] \times SO(n-3)$. The orbit obtained is now (3n-4)-dimensional. The submanifold of all points on the transversal with the coordinates of the type $\mu_1^{\frac{1}{2}} = \mu_2^{\frac{1}{2}} > 0$, $\mu_3^{\frac{1}{2}} > 0$ is two-dimensional so that the configuration space is (3n-2)-dimensional. In quantum mechanical calculations one can neglect such a manifold as one of measure zero to the measure defined on $\mathbb{R}^{3n}$. The same argument excludes the subspaces related to the points on the transversal of the type $\mu_1^{\frac{1}{2}} = \mu_2^{\frac{1}{2}} = \mu_3^{\frac{1}{2}} > 0$.

Before we discuss the orbits in the case that some of $\mu_i^{\frac{1}{2}}$ equal zero, we should give some physical interpretation of the coordinates introduced. The part of the manifold given by $\sigma \in SO(3)$ is one which changes under the simultaneous rotations of all relative coordinates, under the left action of $SO(3)$ with the elements $\tilde{\sigma}$ on ξ. So, the set of $SO(3)$ parameters ω, $\sigma = \sigma(\omega)$ can be chosen to be Euler angles, for example. The transversal coordinates one easily interprets from the relation of the "collective" tensor $\mathcal{T}(\xi)$ to the instantaneous tensor of inertia of the system of n+1 particles

$$I_{ij} = -m\left[\mathcal{T}_{ij} - tr\,\mathcal{T}\,\delta_{ij}\right].$$

The instantaneous principal moments of inertia are

$$I_1 = m\left(\mathring{\xi}_2^2 + \mathring{\xi}_3^2\right)$$

$$I_2 = m\left(\mathring{\xi}_1^2 + \mathring{\xi}_3^2\right)$$

$$I_3 = m\left(\mathring{\xi}_1^2 + \mathring{\xi}_2^2\right).$$

Now we are able to discuss orbits for representative points of the type where some of μ_i's are equal to zero. When one of μ_i's is equal to zero, all particles move in the plane othogonal to the "i" principle axis; when two μ_i's are equal to zero, for example $\mu_1 = \mu_2 = 0$ all particles move on the line orthogonal to the "1" and "2" principal axis; when all μ_i's are equal to zero, all

particles are in the center of mass. All these manifolds in the quantum mechanical case are of measure zero. In reference 24 the authors discuss the $\mathcal{GL}_+(3,\mathbb{R})$ scheme. Here the orbits attached to the representative points of the type where some of μ_i's take the value zero are excluded from the very beginning.
As the final result we get the coordinate transformation

$$\xi_{is} = \sum_{\ell=1}^{3} \sigma_{i\ell}(\omega)\, \mathring{\xi}_\ell\, \sigma'_{\ell s}(\beta).$$

Our new coordinates (for quantum mechanical calculations) are 3 transversal coordinates of the type $\mathring{\xi}_1 > \mathring{\xi}_2 > \mathring{\xi}_3 > 0$ and orbit coordinates which are any set of parameters of the coset space $SO(3)\times SO(n)/H$ where $H = [diag(\mathcal{H}'_3 \times \mathcal{H}'_3)] \times SO(n-3)$ and

$$\mathcal{H}'_3 = \left\{ \begin{pmatrix} 1&0&0\\0&1&0\\0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\0&-1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&1&0\\0&0&-1 \end{pmatrix}, \begin{pmatrix} -1&0&0\\0&-1&0\\0&0&1 \end{pmatrix} \right\}$$

We choose $\sigma(\omega) \in SO(3)/\mathcal{H}'_3$ the left coset space, which gives 3 ω_μ's. The group $\mathcal{H}'_3$ is discrete and do not reduce the number of parameters. Both the $\mathring{\xi}_i$'s and ω_μ's are, as already explained, of collective nature. The coordinates β_τ's $\tau = 1,2,\dots,3n-6$ are the parameters of the right coset space $\sigma'(\beta) \in SO(n)/SO(n-3)$. This decomposition of the point ξ_{is} is unique per construction. The physical nature of coordinates β_τ's will be explained in (II.4), (II.5) and (II.6). There exists one coordinate transformation[16] which brings $\mathring{\xi}_1, \mathring{\xi}_2, \mathring{\xi}_3$ to some other set of coordinates which, in the liquid-drop model approximation, reduces to the well-known β, γ and R_0 coordinates[1].

II.2 The Jacobi matrix and its inverse

In this section we calculate the Jacobi matrix for the change of coordinates $(\xi_{is}) \to (\omega_\mu, \mathring{\xi}_i, \beta_\tau)$. Then we invert it analytically. Before starting with the calculations we explain what all that is good for, and compare our method with the one given in references 24 and 25. Our claim is that we have a method which is clear and simple. We want to determine all possible differential operators in the new coordinates. We do the coordinate transformation in an Euclidean space from Cartesian to transversal and orbit coordinates. We want to see how these coordinates and

corresponding differential operators are related to the laboratory and "body fixed" frame, and to compare it with the "left" and "right" action generators of a Lie group action on certain coset space. Further we want to determine the "velocity field", a classical term, and to find both quantum mechanicaly and classicaly the kinetic energy in the new coordinates. We will also express the invariant volume element in the new coordinates. For all these only the explicit Jacobi matrix and its inverse are needed. In order to be able to determine these quantities we need Lie algebra properties of Θ's, Ψ's, d's and dm's. Those properties are of two types: coordinate independent (given in (I.1) and (I.2)) and coordinate dependent (given in (I.3) for phase space and in (I.4) for configuration space).
We restrict our discussion first to the coordinate transformation in configuration space and determine the transformation of momenta π_{is}, conjugate to ξ_{is}, in differential operators in new coordinates. Given a set of new coordinates

$$(\xi_{is} \mid i=1,2,3 \quad s=1,2,\dots,n) \longrightarrow (\xi_\nu \mid \nu=1,2,\dots,3n)$$

$$\frac{\partial}{\partial \xi_{is}} = \sum_\nu \frac{\partial \xi_\nu}{\partial \xi_{is}} \frac{\partial}{\partial \xi_\nu} =: \sum_\nu {}^t M^{-1}_{is,\nu} \frac{\partial}{\partial \xi_\nu} .$$

For the quantum mechanical kinetic energy one obtains

$$T_{QM} = -\frac{\hbar^2}{2m} \sum_{i,s} \frac{\partial^2}{\partial \xi_{is}^2}$$

$$= -\frac{\hbar^2}{2m} \sum_{i,s} \sum_{\nu=1}^{3n} \sum_{\mu=1}^{3n} {}^t M^{-1}_{is,\nu} \frac{\partial}{\partial \xi_\nu} {}^t M^{-1}_{is,\mu} \frac{\partial}{\partial \xi_\mu} .$$

Of course one can introduce this expression in such a way that it takes the elegant form for the Laplace Beltrami operator in $\mathbb{R}^{3n}$, but one just doesn't need it. Authors in 24 started from the form for the most general Laplace Beltrami operator for an arbitrary Riemmanian manifold expressed in the covariant form

$$\Delta = \frac{1}{(\det g_{\mu\nu})^{\frac{1}{2}}} \frac{\partial}{\partial \xi_\mu} (\det g_{\mu\nu})^{\frac{1}{2}} g^{\mu\nu} \frac{\partial}{\partial \xi_\nu}$$

but already in the next step they switched to Euclidean space where the problem is formulated anyhow.
The invariant volume element in the new coordinates is

$$dm = \prod_{i,s} d\xi_{is} = \det {}^{M}_{g} \prod_{\nu} d\zeta_{\nu} .$$

Let us now consider the classical case. There we also start with the point transformation

$$\zeta_{\nu} = \zeta_{\nu}(\xi_{is})$$

and define new momenta

$$\pi_{\nu} = \sum_{i,s} {}^{t}M_{\nu,is}\, \Pi_{is} .$$

One can easily show that this transformation in phase space is a canonical transformation (contact transformation). One does it by checking the Poisson brackets

$$\{A, B\}(\xi,\pi) = \sum_{i=1}^{3} \sum_{s=1}^{n} \left(\frac{\partial A}{\partial \xi_{is}} \frac{\partial B}{\partial \pi_{is}} - \frac{\partial A}{\partial \pi_{is}} \frac{\partial B}{\partial \xi_{is}} \right)$$

and one obtains that

$$\{\zeta_{\nu}, \pi_{\mu}\}(\xi,\pi) = \delta_{\nu\mu} \; ; \; \{\zeta_{\nu}, \zeta_{\mu}\}(\xi,\pi) = 0 \quad \text{and} \quad \{\pi_{\nu}, \pi_{\mu}\}(\xi,\pi) = 0 .$$

The classical kinetic energy is then given by

$$T_{ke} = \frac{1}{2m} \sum_{i,s} \Pi_{is}^{2}$$

$$= \frac{1}{2m} \sum_{\nu} \sum_{\mu} {}^{t}M^{-1}_{g\, is,\nu}\, {}^{t}M^{-1}_{g\, is,\mu}\, \pi_{\nu} \pi_{\mu} .$$

So it becomes clear that for all our goals we only need M_g and M_g^{-1}. The authors in 24 and 25 use for $\frac{\partial}{\partial \xi_{is}}$ or Π_{is}

$$\Pi_{is} = \sum_{\mu,\nu} \pi_{\mu}^{is}\, g^{\mu\nu}\, \pi_{\nu}$$

and for the kinetic energy an expression proportional to

$$\Delta = \sum_{\mu,\nu} \pi_{\mu}\, g^{\mu\nu}\, \pi_{\nu} - i\hbar \sum_{\mu,\nu} \frac{\partial\, \pi_{\mu}^{is}(\xi)}{\partial \xi_{is}}\, g^{\mu\nu}\, \pi_{\nu}$$

where $\Pi_{\nu} = \sum_{i,s} \pi_{\nu}^{is}(\xi)\, \Pi_{is}$ and $g^{\mu\nu} = (g^{-1})_{\mu\nu}$ with

$$g_{\mu\nu}(\xi) := g(\pi_{\mu}, \pi_{\nu}) = \sum_{i,s} \pi_{\mu}^{is}(\xi)\, \pi_{\nu}^{is}(\xi) .$$

The correspondence with our method can easily be obtained by putting $\pi_\nu^{is} \equiv {}^t\mathcal{Y}_{\nu,is}$ in the case where $\pi_\nu := \frac{\partial}{\partial \xi_\nu}$. It should be mentioned that the authors in 24 start with the linear transformation of momenta

$$\pi_\nu = \sum_{i,s} \pi_\nu^{is}\, \pi_{is} .$$

Our starting point is the coordinate, point transformation in configuration space. Then we determine from it the transformation of the momenta through the chain rule[40] . We find this point of view more convenient for the generalization in phase space where one transforms coordinates in $\mathbb{R}^{6n}$ (see (IV.1) and also references 30 and 41).
The authors in 24 for their calculations invert $g_{\mu\nu}$ and calculated the expression $\frac{\partial \pi_\mu^{is}}{\partial \xi_{is}}$; we invert $\mathcal{Y}$ and calculate $\frac{\partial}{\partial \xi_\nu}\, {}^t\mathcal{Y}^{-1}_{is,\mu}$.
It is clear that the methods are actually equivalent, but we operate only with $\mathcal{Y}$ and $\mathcal{Y}^{-1}$ and don't need $g_{\mu\nu}$ and $g^{\mu\nu}$ (which can be easily determined from $\mathcal{Y}$ and $\mathcal{Y}^{-1}$). It should be stressed that for the determination of the velocity field one must have $\mathcal{Y}^{-1}$. From now on we redefine ${}^t\mathcal{Y}$ and ${}^t\mathcal{Y}^{-1}: {}^t\mathcal{Y} \to \mathcal{Y}$, ${}^t\mathcal{Y}^{-1} \to \mathcal{Y}^{-1}$ in order to avoid the superscript t .

II.2.1 Proposition: The Jacobi matrix for the transition $(\xi_{ls}) \to (\omega_\lambda, \mathring{\xi}_i, \beta_\sigma) \equiv (\xi_\nu \mid \nu = 1,2,\ldots,3n)$ is given through its matrix elements

$$\mathcal{Y}_{\nu,ls} = \frac{\partial \xi_{ls}}{\partial \xi_\nu} \,,$$ which are explicitly given by

$$\text{(I)} \quad \frac{\partial \xi_{ls}}{\partial \omega_\mu} = \sum_{i=1}^{3} \sum_{j=1}^{3} \sigma_{lj}(\omega)\, {}^R\widetilde{\Psi}_{\mu,ji}(\omega)\, \mathring{\xi}_i\, \sigma'_{is}(\beta)$$

$$\text{(II)} \quad \frac{\partial \xi_{ls}}{\partial \mathring{\xi}_w} = \sigma_{lw}(\omega)\, \sigma'_{ws}(\beta)$$

$$\text{(III)} \quad \frac{\partial \xi_{ls}}{\partial \beta_\tau} = \sum_{i=1}^{3} \sum_{t=1}^{n} \sigma_{li}(\omega)\, \mathring{\xi}_i\, {}^L\widetilde{\Psi}_{\tau,it}(\beta)\, \sigma'_{ts}(\beta) .$$

Proof: For (I), (II) and (III) we differentiate the relation

$$\xi_{\ell s} = \sum_{i=1}^{3} \sigma_{\ell i}(\omega)\, \mathring{\xi}_i\, \sigma'_{is}(\beta), \quad \sigma(\omega) \in SO(3)/\mathcal{H}'_3, \quad \sigma'(\beta) \in SO(n)/SO(n-3)$$

with respect to the new coordinates ω, $\mathring{\xi}$ and β, respectively. To determine (I) we use (I.4.4), and determine (III) (I.4.1). □

II.2.2 Proposition: The inverse Jacobi matrix $\mathcal{J}^{-1}$ is given by its matrix elements

$$\mathcal{J}^{-1}_{\omega\ell s,\nu} = \frac{\partial \xi_\nu}{\partial \xi_{\ell s}}$$

$$\text{(I)} \quad \frac{\partial \omega_\mu}{\partial \xi_{\ell s}} = \sum_{\nu=1}^{3} \sum_{g=1}^{3} \sigma_{\ell g}(\omega) \frac{\mathring{\xi}_\nu}{\mathring{\xi}_\nu^2 - \mathring{\xi}_g^2}\, {}^{R}\widetilde{\Theta}_{g\nu,\mu}(\omega)\, \sigma'_{\nu s}(\beta)$$

$$\text{(II)} \quad \frac{\partial \mathring{\xi}_w}{\partial \xi_{\ell s}} = \sigma_{\ell w}(\omega)\, \sigma'_{ws}(\beta)$$

$$\text{(III)} \quad \frac{\partial \beta_\sigma}{\partial \xi_{\ell s}} = \sum_{\nu=1}^{3} \sum_{g=1}^{3} \sigma_{\ell g}(\omega) \frac{\mathring{\xi}_g}{\mathring{\xi}_g^2 - \mathring{\xi}_\nu^2}\, {}^{L}\widetilde{\Theta}_{g\nu,\tau}(\beta)\, \sigma'_{\nu s}(\beta)$$

$$+ \sum_{\nu=4}^{n} \sum_{g=1}^{3} \sigma_{\ell g}(\omega)\, \mathring{\xi}_g^{-1}\, {}^{L}\widetilde{\Theta}_{g\nu,\tau}(\beta)\, \sigma'_{\nu s}(\beta)$$

Proof: In order to determine $\mathcal{J}^{-1}$ we introduce an ansatz for it which contains matrices Θ inverse to Ψ's and undetermined matrices A and B:

$$\text{(I)} \quad \frac{\partial \omega_\mu}{\partial \xi_{\ell s}} = \sum_{\nu=g}^{3} \sum_{g=1}^{3} \sigma_{\ell g}(\omega)\, \frac{1}{2} \left[{}^{R}\widetilde{\Theta}_{g\nu,\mu}(\omega) + A_{g\nu,\mu} \right] \mathring{\xi}_\nu^{-1}\, \sigma'_{\nu s}(\beta)$$

$$- \sum_{\nu=1}^{3} \sum_{g=\nu}^{3} \sigma_{\ell g}(\omega)\, \frac{1}{2} \left[{}^{R}\widetilde{\Theta}_{\nu g,\mu}(\omega) - A_{\nu g,\mu} \right] \mathring{\xi}_\nu^{-1}\, \sigma'_{\nu s}(\beta)$$

$$\text{(II)} \quad \frac{\partial \mathring{\xi}_w}{\partial \xi_{\ell s}} = \sigma_{\ell w}(\omega)\, \sigma'_{ws}(\beta)$$

$$\text{(III)} \quad \frac{\partial \beta_\sigma}{\partial \xi_{\ell s}} = \sum_{g=1}^{3} \sum_{\nu=g}^{n} \sigma_{\ell g}(\omega)\, \mathring{\xi}_g^{-1}\, \frac{1}{2} \left[{}^{L}\widetilde{\Theta}_{g\nu,\tau}(\beta) + B_{g\nu,\tau} \right] \sigma'_{\nu s}(\beta)$$

$$- \sum_{\nu=1}^{3} \sum_{g=\nu}^{3} \sigma_{\ell g}(\omega)\, \mathring{\xi}_g^{-1}\, \frac{1}{2} \left[{}^{L}\widetilde{\Theta}_{\nu g,\tau}(\beta) - B_{\nu g,\tau} \right] \sigma'_{\nu s}(\beta).$$

Now we demand that $\sum_{\varrho=1}^{3}\sum_{s=1}^{n} {}^{\gamma}_{\delta_{\nu,\varrho s}}\,{}^{\gamma^{-1}}_{\varrho s,\mu'} = \delta_{\nu\mu'}$ and using (I.4.6), (I.4.7), (I.2.2) and (I.2.3) after a long but straightforward calculation we get for A and B the following expressions:

$$A_{gt,\mu} = \frac{\overset{\circ}{\xi}{}^2_g + \overset{\circ}{\xi}{}^2_t}{\overset{\circ}{\xi}{}^2_t - \overset{\circ}{\xi}{}^2_g}\ {}^{R}\widetilde{\Theta}_{gt,\mu} \quad \text{for } g\neq t \quad g,t=1,2,3\,,$$

$$A_{gg,\mu} = 0 \quad \text{for } g=1,2,3\,;$$

$$B_{jv,\tau} = \frac{\overset{\circ}{\xi}{}^2_v + \overset{\circ}{\xi}{}^2_j}{\overset{\circ}{\xi}{}^2_j - \overset{\circ}{\xi}{}^2_v}\ {}^{L}\widetilde{\Theta}_{jv,\tau}(\beta) \quad \text{for } j\neq v \quad j,v=1,2,3\,,$$

$$B_{jj,\tau} = 0 \quad \text{for } j=1,2,3\,,$$

$$B_{gt,\tau} = {}^{L}\Theta_{gt,\tau}(\beta) \quad \text{for } g=1,2,3 \text{ and } t=4,5,\dots,n\,. \qquad \square$$

II.3. The generators of the orthogonal groups $SO(3)$ and $SO(n)$ expressed through the new coordinates

Let us first determine the $SO(3)$ left action generators as differential operators acting on functions $F(\xi)$ where $\xi_{is}\in\mathbb{R}^{3n}$.

II.3.1 Definition:

$$F'(\xi) := F({}^{t}\tilde{q}\,\xi)\,, \quad \tilde{q}(\tilde{\gamma})\in SO(3)\,.$$

We use here the exponential parametrization with $\tilde{\gamma}_{iq} = -\tilde{\gamma}_{qi}$. The left action on ξ with the element $\tilde{\sigma}$ in the neighborhood of the identity is given by

$$\xi'_{is} = \sum_{q=1}^{3} {}^{t}\tilde{q}_{iq}\,\xi_{qs} = \xi_{is} - \sum_{q=1}^{3}\tilde{\gamma}_{iq}\,\xi_{qs} + \dots$$

Hence

$$F'(\xi) = F(\xi) + \sum_{i=1}^{3}\sum_{s=1}^{n}(\xi'-\xi)_{is}\frac{\partial}{\partial\xi_{is}}F(\xi)+\dots \approx F(\xi) - \sum_{\substack{i,q=1\\(i<q)}}^{3}\tilde{\gamma}_{iq}\sum_{s=1}^{n}\left(\xi_{qs}\frac{\partial}{\partial\xi_{is}} - \xi_{is}\frac{\partial}{\partial\xi_{qs}}\right)F(\xi)$$

$$=: F(\xi) + \sum_{\substack{i,q=1\\(i<q)}}^{3}\tilde{\gamma}_{iq}\,{}^{L}\chi^{a}_{iq}\,F(\xi)$$

so that

$${}^{L}\chi^{a}_{iq} := \sum_{s=1}^{n}\left(\xi_{is}\frac{\partial}{\partial\xi_{qs}} - \xi_{qs}\frac{\partial}{\partial\xi_{is}}\right).$$

II.3.2 Proposition: The angular momentum operator of the system of $n+1$ particles (hermitian operator),

$$ {}^{L}\hat{\chi}^{a}_{\ell k} := -i\hbar\, {}^{L}\chi^{a}_{\ell k} $$

where ${}^{L}\chi^{a}_{\ell k}$ is as in (II.3.1), acting on the orbit $SO(3)\times SO(n)\big/\mathcal{H}_3' \times SO(n-3)$ takes the form

$$ {}^{L}\hat{\chi}^{a}_{\ell k} = i\hbar \sum_{\mu} {}^{L}\Theta_{\ell k,\mu}(\omega)\frac{\partial}{\partial\omega_\mu}. $$

It can also be written in the form

$$ {}^{L}\hat{\chi}^{a}_{\ell k} = +i\hbar \sum_{g=1}^{3}\sum_{\nu=g+1}^{3} {}^{t}d^{-1}_{\ell k,g\nu}\, L_{g\nu}\,, $$

where $L_{g\nu}$ is defined by

$$ L_{g\nu} := \sum_{\mu=1}^{3} {}^{R}\Theta_{g\nu,\mu}(\omega)\frac{\partial}{\partial\omega_\mu}. $$

Proof: We first calculated the left action $g\ell(3,\mathbb{R})$ generators on $\mathbb{R}^{3n}$

$$ \begin{aligned} {}^{L}\chi_{\ell k} &:= -\sum_{s=1}^{n}\xi_{ks}\frac{\partial}{\partial\xi_{\ell s}}\ {}^{*} \\ &= -\sum_{s=1}^{n}\sum_{\nu=1}^{3n}\xi_{ks}\,\gamma^{-1}_{\ell s,\nu}\frac{\partial}{\partial\xi_\nu} \\ &= -\sum_{g=1}^{3}\sum_{\nu=g+1}^{3}\left(\sigma_{k\nu}(\omega)\sigma_{\ell g}(\omega)\,\mathring{\xi}^{2}_{\nu} + \sigma_{kg}(\omega)\sigma_{\ell\nu}(\omega)\,\mathring{\xi}^{2}_{g}\right)\frac{1}{\mathring{\xi}^{2}_{\nu}-\mathring{\xi}^{2}_{g}}\sum_{\mu=1}^{3}{}^{R}\Theta_{g\nu,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \\ &\quad -\sum_{w=1}^{3}\sigma_{kw}(\omega)\sigma_{\ell w}(\omega)\,\mathring{\xi}_{w}\frac{\partial}{\partial\mathring{\xi}_{w}} \\ &\quad -\sum_{g=1}^{3}\sum_{\nu=g+1}^{3}\left(\sigma_{k\nu}(\omega)\sigma_{\ell g}(\omega)+\sigma_{kg}(\omega)\sigma_{\ell\nu}(\omega)\right)\frac{\mathring{\xi}_{\nu}\mathring{\xi}_{q}}{\mathring{\xi}^{2}_{q}-\mathring{\xi}^{2}_{\nu}}\sum_{\tau=1}^{3n-9}{}^{L}\Theta_{g\nu,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \qquad \text{(II.2.2)} \end{aligned} $$

Then the differential operator ${}^{L}\chi^{a}_{\ell k}$ becomes

$$ \begin{aligned} {}^{L}\chi^{a}_{\ell k} &= -\sum_{g=1}^{3}\sum_{\nu=g+1}^{3}\left(\sigma_{k\nu}(\omega)\sigma_{\ell g}(\omega)-\sigma_{\ell\nu}(\omega)\sigma_{kg}(\omega)\right)\sum_{\mu}{}^{R}\Theta_{g\nu,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \\ &= -\sum_{g=1}^{3}\sum_{\nu=g+1}^{3}{}^{t}d^{-1}_{\ell k,g\nu}(\omega)\sum_{\mu=1}^{3}{}^{R}\Theta_{g\nu,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \qquad \text{(I.4.2)} \\ &= -\sum_{\mu=1}^{3}{}^{L}\Theta_{\ell k,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \qquad \text{(I.1.11)}, \end{aligned} $$

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* ${}^{L}\chi_{\ell k}$ are defined in an analogous way as ${}^{L}\chi^{a}_{\ell k}$ in (II.3.1).

or if we introduce

$$L_{gv} := \sum_{\mu=1}^{3} {}^{R}\oplus_{gv,\mu}(\omega)\frac{\partial}{\partial\omega_\mu}$$

$${}^{L}\chi^{a}_{\ell k} = -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{t}d^{-1}_{\ell k,gv}(\omega)\, L_{gv} \cdot \square$$

In the (II.4) we are going to discuss how the operators L_{gv} are related to the "body fixed" coordinate frame.

II.3.3 Proposition: The classical angular momenta of the system of $n+1$ particles

$${}^{L}_{p}\chi^{a}_{\ell k} = \sum_{s=1}^{n}(\xi_{\ell s}\pi_{ks} - \xi_{ks}\pi_{\ell s}) \qquad \ell<k=1,2,3$$

(a) are the hamiltonian functions* of an action of $SO(3)$ on phase space (ξ_{is},π_{is}), the so-called Poisson action of $SO(3)$ on a symplectic manifold;

(b) the Poisson generators ${}^{L}_{p}\chi^{a}_{\ell k}$ are via the canonical coordinate transformation $(\xi_{is},\pi_{is}) \to (\omega_\mu,\pi_\mu\,;\,\mathring{\xi}_i,\pi_i\,;\,\beta_\tau,\pi_\tau)$ given on the reduced phase space spanned by (ω_μ,π_μ)

$$\begin{aligned}{}^{L}_{p}\chi^{a}_{\ell k} &= -\sum_{\mu=1}^{3} {}^{L}\oplus_{\ell k,\mu}(\omega)\pi_\mu \\ &= -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{t}d^{-1}_{\ell k,gv}(\omega)\, L_{gv}\end{aligned}$$

where the π_μ's are the canonically conjugate momenta to ω_μ's , and

$$L_{gv} := \sum_{\mu=1}^{3} {}^{R}\oplus_{gv,\mu}(\omega)\pi_\mu \qquad g<v=1,2,3.$$

Proof: (a) We introduce ${}^{L}_{p}\chi^{a}_{\alpha}$ where $\alpha \leftrightarrow \ell k$ and $\alpha\ell k$ is always the cyclic permutation of 1 2 3. Then we calculate Poisson brackets of ${}^{L}_{p}\chi^{a}_{\alpha}$ in Euclidean phase space

$$\left\{{}^{L}_{p}\chi^{a}_{\alpha}, {}^{L}_{p}\chi^{a}_{\beta}\right\}(\xi,\pi) = \sum_{\delta}\varepsilon_{\alpha\beta\delta}\,{}^{L}_{p}\chi^{a}_{\delta}$$

and through that find that $-{}^{L}_{p}\chi^{a}_{\alpha}$ with respect to $\{\ ,\ \}(\xi,\pi)$ give the realisation of the abstract $SO(3)$ group with generators L_ν

*Hamiltonian functions of the Poisson action we are going to call Poisson operators.

$[L_\alpha, L_\beta] = -\sum_\gamma \varepsilon_{\alpha\beta\gamma} L_\gamma$ (note that ${}^L\chi^a_{ek}$ in (II.3.2) fulfil these commutation relations). We define the Poisson action of $SO(3)$ on phase space as homomorphism (isomorphism) $(L_\alpha) \leftrightarrow (-{}^L_p\chi^a_\alpha)$, $H_{L_\alpha} \equiv -{}^L_p\chi^a_\alpha$ (see ref. 42 p. 371).

(b) Here we proceed analogously to (II.3.2) and get that ${}^L_p\chi^a_{ek} = -\sum_\mu {}^L\Theta_{ek,\mu}(\omega)\pi_\mu$. From the fact that the coordinate transformation $(\xi_{is}, \pi_{is}) \to (\omega_\nu, \pi_\nu; \overset{\circ}{\xi}_i, \pi_i; \beta_\tau, \pi_\tau)$ in Euclidean phase space is a canonical transformation we find what we already expected

$$\left\{\sum_\mu {}^L\Theta_{\alpha\mu}(\omega)\pi_\mu, \sum_\nu {}^L\Theta_{\beta\nu}(\omega)\pi_\nu\right\}(\omega_\nu \overset{\circ}{\xi}_i \beta_\tau, \pi_\nu \pi_i \pi_\tau) = \left\{\sum_\mu {}^L\Theta_{\alpha\mu}(\omega)\pi_\mu, \sum_\nu {}^L\Theta_{\beta\nu}(\omega)\pi_\nu\right\}(\omega_\nu, \pi_\nu)$$

$$= -\sum_\gamma \varepsilon_{\alpha\beta\gamma} \sum_\mu {}^L\Theta_{\gamma\mu}(\omega)\pi_\mu . \quad \square$$

Here we want to stress that there exists the correlation between the Poisson generators H_{L_α} and the differential operators of the group on phase space

$$\{H_{L_\alpha}, F\}(\xi,\pi) = D_{L_\alpha} F(\xi,\pi)$$

what is in the case of $SO(3)$

$$\left\{-{}^L_p\chi^a_{ek}, F\right\}(\xi,\pi) =$$

$$= \sum_{s=1}^n \left(\xi_{es}\frac{\partial}{\partial \xi_{ks}} - \xi_{ks}\frac{\partial}{\partial \xi_{es}}\right)F + \sum_{s=1}^n \left(\pi_{es}\frac{\partial}{\partial \pi_{ks}} - \pi_{ks}\frac{\partial}{\partial \pi_{es}}\right)F .$$

In the special case where we consider only the space of functions on a configuration space $F \equiv F(\xi)$, D_{L_α} reduce to the ${}^L\chi^a_{ek}$.

II.3.4 Definition: Here we introduce the SO(n) right action generators as differential operators acting on functions $F(\xi)$ where $\xi_{is} \in \mathbb{R}^{3n}$.

$$F^1(\xi) := F(\xi\tilde{q}) , \quad \tilde{q}(\tilde{\gamma}) \in SO(n).$$

We use the exponential parametrization with $\tilde{\gamma}_{ie} = -\tilde{\gamma}_{ei}$. The right action on ξ with the element $\tilde{q}$ in the neighborhood of the identity is given by

$$\xi'_{is} = \sum_{t=1}^n \xi_{it}\tilde{q}_{ts} = \xi_{is} + \sum_{t=1}^n \xi_{it}\tilde{\gamma}_{ts} + \cdots .$$

Hence

$$F'(\xi) = F(\xi) + \sum_{i=1}^{3}\sum_{s=1}^{n} (\xi'-\xi)_{is}\frac{\partial}{\partial \xi_{is}} F(\xi)$$

$$\approx F(\xi) + \sum_{\substack{t,s=1\\(t<s)}}^{n} \tilde{\gamma}_{ts} \sum_{i=1}^{3} \left(\xi_{it}\frac{\partial}{\partial \xi_{is}} - \xi_{is}\frac{\partial}{\partial \xi_{it}}\right) F(\xi)$$

$$=: F(\xi) + \sum_{\substack{t,s=1\\(t<s)}}^{n} \tilde{\gamma}_{ts}\, {}^{R}\chi^{a}_{ts}\, F(\xi)$$

so that $${}^{R}\chi^{a}_{ts} := \sum_{i=1}^{3} \left(\xi_{it}\frac{\partial}{\partial \xi_{is}} - \xi_{is}\frac{\partial}{\partial \xi_{it}}\right).$$

II.3.5 Proposition: The right action generators of SO(n) acting on the orbit $SO(3)\times SO(n)/\mathcal{H}_3' \times SO(n-3)$, expressed through the new ω and β coordinates are

$${}^{R}X^{a}_{st} = -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{t}d_{st,gv}(\beta)\mathcal{L}_{gv} - \sum_{g=1}^{3}\sum_{v=4}^{n} {}^{t}d_{st,gv}(\beta)\,\phi_{gv}$$

where $\mathcal{L}_{gv}$ and ϕ_{gv} are defined as

$$\mathcal{L}_{gv} := -\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial \beta_\tau}, \quad g<v = 1,2,3 \quad \text{(for } \tau \text{ see convention at the beginning of the section I.2)},$$

$$\phi_{gv} := -\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial \beta_\tau}, \quad g=1,2,3 \quad v=4,5,\dots,n,$$

and the matrix d is givne in (I.4.2).
The generators ${}^{R}X^{a}_{st}$ can also be written

$${}^{R}X^{a}_{st} = \sum_{\tau=1}^{3n-6} {}^{R}\Theta_{st,\tau}(\beta)\frac{\partial}{\partial \beta_\tau},$$

which means that the action of SO(n) on the whole orbit manifold reduces to the action on the right coset space what we have already expected. The hermitean operator can be written

$${}^R\hat{X}^a_{st} := -i\hbar\, {}^RX^a_{st}.$$

Proof: We first calculate the right action $GL(n,\mathbb{R})$ generators

$$\begin{aligned}
{}^RX_{st} &= \sum_{\ell=1}^{3} \xi_{\ell s}\frac{\partial}{\partial \xi_{\ell t}} \\
&= \sum_{\ell=1}^{3} \xi_{\ell s}\, M^{-1}_{\ell t,\nu}\frac{\partial}{\partial \xi_\nu} \\
&= \sum_{g=1}^{3}\sum_{v=g+1}^{3}\left(\sigma'_{gs}(\beta)\sigma'_{vt}(\beta)+\sigma'_{vs}(\beta)\sigma'_{gt}(\beta)\right)\frac{\overset{\circ}{\xi}_g\overset{\circ}{\xi}_v}{\overset{\circ}{\xi}{}^2_v-\overset{\circ}{\xi}{}^2_g}\sum_{\mu=1}^{3}{}^R\Theta_{gv,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \\
&+ \sum_{w=1}^{3}\sigma'_{ws}(\beta)\sigma'_{wt}(\beta)\,\overset{\circ}{\xi}_w\frac{\partial}{\partial\overset{\circ}{\xi}_w} \\
&+ \sum_{g=1}^{3}\sum_{v=g+1}^{3}\left(\sigma'_{gs}(\beta)\sigma'_{vt}(\beta)\overset{\circ}{\xi}_g+\sigma'_{vs}(\beta)\sigma'_{gt}(\beta)\overset{\circ}{\xi}_v\right)\frac{1}{\overset{\circ}{\xi}{}^2_g-\overset{\circ}{\xi}{}^2_v}\sum_{\tau=1}^{3n-6}{}^L\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \\
&+ \sum_{g=1}^{3}\sum_{v=4}^{n}\sigma'_{gs}(\beta)\sigma'_{vt}(\beta)\sum_{\tau=1}^{3n-6}{}^L\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \qquad \text{(II.2.2).}
\end{aligned}$$

Then the right action SO(n) generators are

$$\begin{aligned}
{}^RX^a_{st} &= \sum_{g=1}^{3}\sum_{v=g+1}^{n}\left(\sigma'_{gs}(\beta)\sigma'_{vt}(\beta)-\sigma'_{gt}(\beta)\sigma'_{vs}(\beta)\right)\sum_{\tau=1}^{3n-6}{}^L\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \\
&= \sum_{g=1}^{3}\sum_{v=g+1}^{n}{}^td_{st,gv}(\beta)\sum_{\tau=1}^{3n-6}{}^L\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \qquad \text{(I.4.2)} \\
&+ \sum_{\tau=1}^{3n-6}{}^R\Theta_{st,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \qquad \text{(I.1.11), (I.2.2)}
\end{aligned}$$

or if we introduce $\mathcal{L}_{gv} := -\sum_{\tau=1}^{3n-6}{}^L\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau}$ for $g<v=1,2,3$ and

$$\phi_{gv} := -\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{gv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau} \quad \text{for} \quad g=1,2,3 \quad v=4,5,\ldots,n$$

$${}^{R}X^{a}_{st} = -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{t}d_{st,gv}(\beta)\mathcal{L}_{gv} - \sum_{g=1}^{3}\sum_{v=4}^{n} {}^{t}d_{st,gv}(\beta)\,\phi_{gv}. \quad \square$$

Note that $\mathcal{L}_{g<v}$ g,v = 1,2,3, although defined to include in general all partial derivatives w.r. t. β's (β's parametrize the right coset space SO(n)/SO(n-3)), through some particular choice of β's can be seen as the left action generators on SO(3) submanifold of the manifold SO(n)/SO(n-3). The $\phi_{g<v}$ g = 1,2,3 v = 4,5,...n include partial derivatives w.r. t. all β's : with some choice of coordinates β such that β_1, β_2 and β_3 parametrize SO(3) submanifold one does not automatically obtain $\phi_{g<v}$ defined on some submanifold parametrized by $\beta_4, \ldots, \beta_{3n-6}$! This property of ϕ's will be particularly discussed in chapter III. It should also be said that ϕ's are not some left action generators, because in general on the right coset space the left action is not defined.

In (II.4) we are going to discuss how the operators $\mathcal{L}_{gv}$ and ϕ_{gv} are related to the "body fixed" coordinate frame. We state now the proposition II.3.6 analogous to II.3.5 in the classical case and in that sense also analogous to proposition II.3.3.

II.3.6 Proposition: The classical functions

$${}^{R}_{p}X^{a}_{st} := \sum_{i=1}^{3}\left(\xi_{is}\pi_{it} - \xi_{it}\pi_{is}\right) \qquad s<t=1,2,\ldots,n$$

(a) are the SO(n) Poisson generators on phase space

(b) Hamiltonian functions ${}^{R}_{p}X^{a}_{st}$ are via canonical coordinate transformation $(\xi_{is},\pi_{is}) \to (\omega_\mu,\pi_\mu;\ \xi_i,\pi_i;\ \beta_\tau,\pi_\tau)$ given on the reduced phase space spanned by (β_τ,π_τ)

$${}^{R}_{p}X^{a}_{st} = \sum_{\tau=1}^{3n-6} {}^{R}\Theta_{st,\tau}(\beta)\pi_\tau$$

$$= -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{t}d_{st,gv}(\beta)\mathcal{L}_{gv} - \sum_{g=1}^{3}\sum_{v=4}^{n} {}^{t}d_{st,gv}(\beta)\,\phi_{gv}$$

where the π_τ's are the canonically conjugate momenta to β_τ's and

$$\mathcal{L}_{gv} := -\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{gv,\tau}(\beta)\,\pi_\tau \quad ; \quad g<v=1,2,3 \;,$$

$$\phi_{gv} := -\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{gv,\tau}(\beta)\,\pi_\tau, \quad g=1,2,3 \quad v=4,5,\ldots,n. \;\square$$

II.4. The kinetic energy

In this section we are going to calculate explicitly the kinetic energy in the quantum mechanical and classical case using the result for H_g^{-1} for the coordinate transformation

$$(\xi_{is} \mid i=1,2,3 \quad s=1,2,\ldots,n) \to (\omega_\mu, \overset{\circ}{\xi}_w, \beta_\tau \mid \mu=1,2,3 \quad w=1,2,3 \quad \tau=1,2,\cdots,3n-6).$$

II.4.1 Proposition: The quantum mechanical kinetic energy on $\mathbb{R}^{3n}$ is given in the new set of coordinates $(\omega_\mu, \overset{\circ}{\xi}_w, \beta_\tau)$ by

$$T = \frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{\overset{\circ}{\xi}{}^2_v + \overset{\circ}{\xi}{}^2_g}{(\overset{\circ}{\xi}{}^2_v - \overset{\circ}{\xi}{}^2_g)^2}\,\hat{L}^2_{gv}$$

$$+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{\overset{\circ}{\xi}{}^2_v + \overset{\circ}{\xi}{}^2_g}{(\overset{\circ}{\xi}{}^2_v - \overset{\circ}{\xi}{}^2_g)^2}\,\hat{\mathcal{L}}^2_{gv}$$

$$+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{4\,\overset{\circ}{\xi}_v\overset{\circ}{\xi}_g}{(\overset{\circ}{\xi}{}^2_v - \overset{\circ}{\xi}{}^2_g)^2}\,\hat{L}_{gv}\hat{\mathcal{L}}_{gv}$$

$$+\frac{1}{2m}\sum_{v=1}^{3}\hat{t}{}^2_v$$

$$-\frac{ih}{2m}\left\{\sum_{v=1}^{3}\frac{n-3}{\overset{\circ}{\xi}_v}\,\hat{t}_v + \sum_{v=1}^{3}\sum_{\substack{g=1\\(v\neq g)}}^{3}\frac{2\overset{\circ}{\xi}_v}{\overset{\circ}{\xi}{}^2_v - \overset{\circ}{\xi}{}^2_g}\,\hat{t}_v\right\}$$

$$+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=4}^{n}\overset{\circ}{\xi}{}^{-2}_g\,\hat{\phi}^2_{gv}$$

where the hermitian differential operators $\hat{L}_{g\nu}$, $\hat{\mathcal{L}}_{g\nu}$ and $\hat{\Phi}_{g\nu}$ are the redefined operators with respect to L, $\mathcal{L}$ and Φ introduced in (II.3.2) and (II.3.5) so that they are the operators from (II.3.2) and (II.3.5) multiplied by $-i\hbar$. The differential operators $\hat{t}_\nu := -i\hbar \frac{\partial}{\partial \xi^\circ_\nu}$.

Proof: The proof is somewhat longer but straightforward. We give it here in order to demonstrate how the Lie-algebraic tools developed in (I.1), (I.2) and (I.4) can be used in practical calculations.

As already stated in (II.2)

$$T = -\frac{\hbar^2}{2m} \sum_{i=1}^{3} \sum_{s=1}^{n} \frac{\partial}{\partial \xi_{is}} \frac{\partial}{\partial \xi_{is}}$$

$$= -\frac{\hbar^2}{2m} \sum_{\nu=1}^{3n} \sum_{\nu'=1}^{3n} \sum_{i=1}^{3} \sum_{s=1}^{n} \gamma^{-1}_{is,\nu} \frac{\partial}{\partial \xi_\nu} \gamma^{-1}_{is,\nu'} \frac{\partial}{\partial \xi_{\nu'}} .$$

Now we set $(\xi_\nu) \equiv (\omega_\mu, \overset{\circ}{\xi}_w, \beta_\tau)$ and insert for γ^{-1} expression obtained in (II.2.2)

$$\sum_{\mu=1}^{3} \sum_{\mu'=1}^{3} \sum_{i=1}^{3} \sum_{s=1}^{n} \gamma^{-1}_{is,\mu} \frac{\partial}{\partial \omega_\mu} \gamma^{-1}_{is,\mu'} \frac{\partial}{\partial \omega_{\mu'}} =$$

$$= \sum_{k=1}^{3} \sum_{\nu=k+1}^{3} \sum_{g=1}^{\nu-1} \frac{\overset{\circ}{\xi}^2_\nu}{(\overset{\circ}{\xi}^2_\nu - \overset{\circ}{\xi}^2_k)(\overset{\circ}{\xi}^2_\nu - \overset{\circ}{\xi}^2_g)} \sum_{\mu=1}^{3} {}^R\Theta_{k\nu,\mu}(\omega)\, {}^R\tilde{\mathcal{L}}_{\mu,kg}(\omega) \sum_{\mu'=1}^{3} {}^R\Theta_{g\nu,\mu'}(\omega) \frac{\partial}{\partial}$$

$$- \frac{1}{\hbar^2} \sum_{g=1}^{3} \sum_{\nu=g+1}^{3} \frac{\overset{\circ}{\xi}^2_\nu + \overset{\circ}{\xi}^2_g}{(\overset{\circ}{\xi}^2_\nu - \overset{\circ}{\xi}^2_g)^2} \hat{L}^2_{g\nu} \qquad \text{(I.4.5), (II.3.2)}$$

$$= -\frac{1}{\hbar^2} \sum_{g=1}^{3} \sum_{\nu=g+1}^{3} \frac{\overset{\circ}{\xi}^2_\nu + \overset{\circ}{\xi}^2_g}{(\overset{\circ}{\xi}^2_\nu - \overset{\circ}{\xi}^2_g)^2} \hat{L}^2_{g\nu} \qquad \text{(I.1.13)}$$

$$\sum_{w=1}^{3} \sum_{w'=1}^{3} \sum_{i=1}^{3} \sum_{s=1}^{n} \gamma^{-1}_{is,w} \frac{\partial}{\partial \overset{\circ}{\xi}_w} \gamma^{-1}_{is,w'} \frac{\partial}{\partial \overset{\circ}{\xi}_{w'}} = \sum_{w=1}^{3} \frac{\partial^2}{\partial \overset{\circ}{\xi}^2_w}$$

$$=: -\frac{1}{\hbar^2} \sum_{w=1}^{3} \hat{t}^2_w \; ;$$

$$\sum_{\tau=1}^{3n-6}\sum_{\tau'=1}^{3n-6}\sum_{i=1}^{3}\sum_{s=1}^{n} \mathcal{Y}^{-1}_{is,\tau}\frac{\partial}{\partial\beta_\tau}\mathcal{Y}^{-1}_{is,\tau'}\frac{\partial}{\partial\beta_{\tau'}} =$$

$$= -\frac{1}{\hbar^2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{\mathring{\xi}^2_g - \mathring{\xi}^2_v}{(\mathring{\xi}^2_g - \mathring{\xi}^2_v)^2}\hat{\mathcal{L}}^2_{gv}$$

$$-\frac{1}{\hbar^2}\sum_{g=1}^{3}\sum_{v=4}^{n}\mathring{\xi}^{-2}_g\,\hat{\mathcal{F}}^2_{gv} \qquad \text{(I.4.1), (II.3.5);}$$

$$\sum_{\mu=1}^{3}\sum_{\tau=1}^{3n-6}\sum_{i=1}^{3}\sum_{s=1}^{n} \mathcal{Y}^{-1}_{is,\mu}\frac{\partial}{\partial\omega_\mu}\mathcal{Y}^{-1}_{is,\tau}\frac{\partial}{\partial\beta_\tau} =$$

$$= \sum_{g=1}^{3}\sum_{v=g+1}^{3}\sum_{k=1}^{v-1}\sum_{\mu=1}^{3} {}^{R}\Theta_{gv,\mu}(\omega)\,{}^{R}\tilde{\mathcal{F}}_{\mu,gk}(\omega)\frac{\mathring{\xi}_v}{\mathring{\xi}^2_v - \mathring{\xi}^2_g}\frac{\mathring{\xi}_k}{\mathring{\xi}^2_k - \mathring{\xi}^2_v}\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{kv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau}$$

$$-\frac{1}{\hbar^2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{2\,\mathring{\xi}_v\mathring{\xi}_g}{(\mathring{\xi}^2_v - \mathring{\xi}^2_g)^2}\hat{\mathcal{L}}_{gv}\hat{\mathcal{L}}_{gv} \qquad \text{(I.4.5), (II.3.2), (II.3.5)}$$

$$= -\frac{1}{\hbar^2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{2\,\mathring{\xi}_v\mathring{\xi}_g}{(\mathring{\xi}^2_v - \mathring{\xi}^2_g)^2}\hat{\mathcal{L}}_{gv}\hat{\mathcal{L}}_{gv} \qquad \text{(I.1.13);}$$

$$\sum_{\tau=1}^{3n-6}\sum_{\mu=1}^{3}\sum_{i=1}^{3}\sum_{s=1}^{n} \mathcal{Y}^{-1}_{is,\tau}\frac{\partial}{\partial\beta_\tau}\mathcal{Y}^{-1}_{is,\mu}\frac{\partial}{\partial\omega_\mu} =$$

$$= -\frac{1}{\hbar^2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{2\,\mathring{\xi}_v\mathring{\xi}_g}{(\mathring{\xi}^2_v - \mathring{\xi}^2_g)^2}\hat{\mathcal{L}}_{gv}\hat{\mathcal{L}}_{gv} \qquad \text{(I.4.1), (II.3.2), (II.3.5), (I.2.3);}$$

$$\sum_{\mu=1}^{3}\sum_{w=1}^{3}\sum_{i=1}^{3}\sum_{s=1}^{n} \mathcal{Y}^{-1}_{is,\mu}\frac{\partial}{\partial\omega_\mu}\mathcal{Y}^{-1}_{is,w}\frac{\partial}{\partial\mathring{\xi}_w} =$$

$$= \frac{i}{\hbar}\sum_{v=1}^{3}\sum_{g=v+1}^{3}\frac{\mathring{\xi}_v}{\mathring{\xi}^2_v - \mathring{\xi}^2_g}\hat{t}_v \qquad \text{(I.4.1), (I.1.13);}$$

$$\sum_{w=1}^{3}\sum_{\mu=1}^{3}\sum_{i=1}^{3}\sum_{s=1}^{n} \mathcal{J}^{-1}_{is,w}\frac{\partial}{\partial \xi_w}\,\mathcal{J}^{-1}_{is,\mu}\frac{\partial}{\partial \omega_\mu} = 0\,;$$

$$\sum_{w=1}^{3}\sum_{\tau=1}^{3n-6}\sum_{i=1}^{3}\sum_{s=1}^{n} \mathcal{J}^{-1}_{is,w}\frac{\partial}{\partial \xi_w}\,\mathcal{J}^{-1}_{is,\tau}\frac{\partial}{\partial \beta_\tau} = 0\,;$$

$$\sum_{\tau=1}^{3n-6}\sum_{w=1}^{3}\sum_{i=1}^{3}\sum_{s=1}^{n} \mathcal{J}^{-1}_{is,\tau}\frac{\partial}{\partial \beta_\tau}\,\mathcal{J}^{-1}_{is,w}\frac{\partial}{\partial \xi_w} =$$

$$= \frac{i}{\hbar}\sum_{v=1}^{3}\sum_{q=v+1}^{3}\frac{\xi_v}{\xi_v^2-\xi_q^2}\,\hat{\mathcal{L}}_v + \frac{i}{\hbar}\sum_{v=1}^{3}\frac{n-3}{\xi_v}\,\hat{\mathcal{L}}_v \qquad \text{(I.4.5), (I.2.3).}$$

The expression for the kinetic energy evaluated in (II.4.1) is an exact result which has been obtained by many authors[21,22,23,25] and here also. Its good features are, that one can recognize in it certain terms which were proposed from different authors as phenomenological collective energy parts, let us mention only Bohr and Mottelson[1,2].

Now we are going to compare our result and notation to those given in ref. 23 and 25. First of all we defined the hermitean operators $\hat{\mathcal{L}}$'s and $\hat{\phi}$'s as the sum of partial derivatives w.r. t. all β's, parameters of the right coset space $SO(n)/SO(n-3) \equiv \mathcal{C}^R$ (see (II.3.5)). It is rather trivial to state that $\hat{\mathcal{L}}$'s can be chosen as left action generators on SO(3) submanifold of $\mathcal{C}^R$. As far as $\hat{\phi}$'s are concerned, we claim that they are in $SO(3)\times SO(n)$ scheme intrinsic only w.r. t. Vangas' collectivity and not w.r.t. $gl_+(3,\mathbb{R})$ collectivity: they contain the partial derivatives w.r. t. SO(3) collective parameters too.

In the reference 23 one finds the following terms: laboratory frame "body-fixed" or "intrinsic" frame, reference frame in particle label space, the "new" reference frame in particle label space and the so-called "internal subspace". There are also two terms in reference 43 p.251: "in" the frame and "referred" to the frame. "Body-fixed" frame is defined by its axes given in laboratory frame by the columns of $\sigma(\omega)$, $\{(\sigma_{1i}\,\sigma_{2i}\,\sigma_{3i})\,|\,i=1,2,3\}$ (one defines

it by diagonalization of the "collective tensor" $\mathcal{T}(\xi)$); the "new" reference frame in particle label space is defined by its axes given in the particle label space by the rows of $\sigma'(\beta)$, $\{(\sigma'_{s1}\,\sigma'_{s2}\,\dots\,\sigma'_{sn}) \mid s=1,2,\dots,n\}$ (one finds it by diagonalization of the "intrinsic tensor" $\mathcal{T}(\xi)$); and finally the "internal" subspace is spanned by the first three rows of $\sigma'(\beta)$, $\{(\sigma'_{s1}\,\sigma'_{s2}\,\dots\,\sigma'_{sn}) \mid s=1,2,3\}$. From the laboratory frame one vector is brought to its form "referred" to the frame via $SO(3)$ rotation $\sigma(\omega)$ which diagonalizes $\mathcal{T}(\xi)$. A certain vague link had been made between the "body-fixed" frame and the "new" particle label frame through the fact that the $\mathcal{T}(\xi)$ tensor via an SO(n) transformation $\sigma'(\beta)$ would be brought to the identical diagonal form as $\mathcal{T}(\xi)$ (see ref. 23 p.216) If one takes it for granted then the "new" internal subspace and particularly its "internal subspace" becomes the space where one describes additional motion of the system w.r. t. its instantaneous main axes of inertia tensor, the so-called motion "in" the body-fixed frame. Our hermitian operators $\hat{\mathcal{L}}$ and $\hat{\mathcal{J}}$ could be seen as defined in the "new" references frame in particle label space, in the body [43]. The left action generators $\hat{\mathcal{L}}$'s can be identified by the vortex spin [15] which plays the role of $\vec{J}$ of Bohr and Mottelson. $\hat{\mathcal{L}}$'s fulfil the same commutation relations as ${}^{L}\hat{\chi}^{a}$'s in the laboratory frame. The hermitian right action generators $\hat{L}$ are related to the angular momentum operator in the laboratory frame

$$\hat{L}_{g<\nu} = -\sum_{\mu=1}^{3} {}^{t}d_{g<\nu,\mu}(\omega)\,{}^{L}\hat{\chi}^{a}_{\mu} \qquad \text{(II.3.2).}$$

One sees that the transformation from the left action generators to the right action generators is the rotation ${}^{t}d(\omega)$ (see (I.4.3) multiplied by -1. The expression $\sum_{\mu=1}^{3} {}^{t}d_{g<\nu,\mu}(\omega)\,{}^{L}\hat{\chi}^{a}_{\mu}$ determines clearly projections of the angular momentum vector operator [44] to the "body-fixed" axes

$$\hat{\hat{L}}_{g<\nu} := \sum_{\mu=1}^{3} {}^{t}d_{g<\nu,\mu}(\omega)\,{}^{L}\hat{\chi}^{a}_{\mu} = -\hat{L}_{g<\nu}$$

so that $\hat{\hat{L}}_{g<\nu}$ is the angular momentum "referred" to the body-fixed frame [43]. Operators ${}^{L}\hat{\chi}^{a}_{g<\nu}$ and $\hat{L}_{g<\nu}$ as the left and the

right action generators fulfil the same commutation relations (see (I.1.18)). Then $\overleftarrow{L}_{q<\nu}$ fulfil the commutation relations with the changed sign. If we in (II.4.1) set $-\overleftarrow{L}$ instead of $\hat{L}$, our result becomes identical to one in ref. 25. Let us write again the expression for the kinetic energy replacing $\hat{L}$ by $-\overleftarrow{L}$ and $\hat{\mathcal{L}}$ by $\overrightarrow{\mathcal{J}}$

$$
\begin{aligned}
T = & \frac{1}{2m} \sum_{q=1}^{3} \sum_{\nu=q+1}^{3} \frac{\mathring{\xi}_\nu^2 + \mathring{\xi}_q^2}{(\mathring{\xi}_\nu^2 - \mathring{\xi}_q^2)^2} \overleftarrow{L}_{q\nu}^2 \\
& + \frac{1}{2m} \sum_{q=1}^{3} \sum_{\nu=q+1}^{3} \frac{\mathring{\xi}_\nu^2 + \mathring{\xi}_q^2}{(\mathring{\xi}_\nu^2 - \mathring{\xi}_q^2)^2} \mathcal{J}_{q\nu}^2 \\
& - \frac{1}{2m} \sum_{q=1}^{3} \sum_{\nu=q+1}^{3} \frac{4\mathring{\xi}_\nu \mathring{\xi}_q}{(\mathring{\xi}_\nu^2 - \mathring{\xi}_q^2)^2} \overleftarrow{L}_{q\nu} \mathcal{J}_{q\nu} + T_{\text{vibrational}} \\
& + \frac{1}{2m} \sum_{q=1}^{3} \sum_{\nu=4}^{n} \mathring{\xi}_q^{-2} \overrightarrow{\phi}_{q\nu}^2 .
\end{aligned}
$$

As we already stated, $SO(3)\times SO(n)$ scheme is particularly convenient for Vanagas' SO(n) collectivity. Actually, Vanagas introduced this definition of collectivity for the manifold having in mind work of Zickendraht [21] and Dzyublik et al. [22] which were, as stated in the general introduction of this work, coordinate transformations including $SO(3)$ and SO(n) group. W.r.t. this criterion, the first and fourth lines in (II.4.1) define the collective kinetic energy, the seond and sixth, the intrinsic, and the third, coupling or mixing term. It is clear that this criterion for collectivity considers vorticities as intrinsic. Stated differently, that would be some irrotationa collectivity. As far as kinetic energy is concerned, it makes much more sense to introduce $g\mathcal{L}_+(3,\mathbb{R})$ criterion for collectivity. But our coordinates chosen in $SO(3)\times SO(n)$ scheme are not adapted for $g\mathcal{L}_+(3,\mathbb{R})$ collectivity! Authors in 23, 25 and 26 claim that it is evident that three generators $\hat{L}$ (or $\overleftarrow{L}$) plus three $\hat{t}$'s plus three $\hat{\mathcal{L}}$'s are $g\mathcal{L}_+(3,\mathbb{R})$ generators, and from that point of view the first four lines in (II.4.1) define the collective $g\mathcal{L}_+(3,\mathbb{R})$ part and the last line is purely intrinsic. We agree with the first part of the statement, because as we already stated it

is really possible to choose β's so that $\hat{\mathcal{L}}$ becomes defined only on the SO(3) submanifold of the right coset space SO(n)/SO(n-3). But the second part of the statement, that the last line in the expression (II.4.1) is purely intrinsic w.r. t. $g\mathcal{L}_+(3,\mathbb{R})$ collectivity can not be stated: it is not true that $\hat{\phi}$'s can be seen as differential operators which do not contain partial derivatives w.r. t. the SO(3) collective parameters without changing the complete expression for the kinetic energy (see chapter III). In order to see whether kinetic energy for A particles really separates into two parts such that the first part includes vector fields defined only on the collective manifold spanned with any set of $g\mathcal{L}_+(3,\mathbb{R})$ parameters, and the second part which includes only differential operators defined on some complementary (3n-9)-dimensional submanifold, one should introduce such a coordinate scheme which actually identifies those two different submanifolds. In our opinion the $g\mathcal{L}_+(3,\mathbb{R})\times SO(n)$ scheme is exactly the one which would be good for this test, and we are going to develop it in chapter III.

II.4.2 Proposition: The classical kinetic energy given in the new set of phase space canonical coordinates $(\omega_\mu, \pi_\mu ; \mathring{\xi}_i, \pi_i ; \beta_\tau, \pi_\tau)$ is

$$
\begin{aligned}
T_{kl} = {} & \frac{1}{2m} \sum_{q=1}^{3} \sum_{v=q+1}^{3} \frac{\mathring{\xi}_v^2 + \mathring{\xi}_q^2}{(\mathring{\xi}_v^2 - \mathring{\xi}_q^2)^2} L_{qv}^2 \\
& + \frac{1}{2m} \sum_{q=1}^{3} \sum_{v=q+1}^{3} \frac{\mathring{\xi}_v^2 + \mathring{\xi}_q^2}{(\mathring{\xi}_v^2 - \mathring{\xi}_q^2)^2} \mathcal{L}_{qv}^2 \\
& + \frac{1}{2m} \sum_{q=1}^{3} \sum_{v=q+1}^{3} \frac{4\,\mathring{\xi}_v \mathring{\xi}_q}{(\mathring{\xi}_v^2 - \mathring{\xi}_q^2)^2} L_{qv} \mathcal{L}_{qv} \\
& + \frac{1}{2m} \sum_{w=1}^{3} \pi_w^2 \\
& + \frac{1}{2m} \sum_{q=1}^{3} \sum_{v=4}^{n} \mathring{\xi}_q^{-2} \hat{\phi}_{qv}^2
\end{aligned}
$$

where $L_{gv} = \sum_{\mu=1}^{3} {}^{R}\Theta_{gv,\mu}(\omega)\,\pi_{\mu} \qquad g<v=1,2,3$

$$\mathcal{L}_{gv} = -\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{gv,\tau}(\beta)\,\pi_{\tau} \qquad g<v=1,2,3$$

$$\phi_{gv} = -\sum_{\tau=1}^{3n-6} {}^{L}\Theta_{gv,\tau}(\beta)\,\pi_{\tau} \qquad g=1,2,3 \quad v=4,5,\ldots,n.$$

Proof: Setting the expression for $\overset{\circ}{M}{}^{-1}_{\beta 3}$ in (II.2.2) in the expression for T_{kl}, $T_{kl} = \frac{1}{2m}\sum_{i=1}^{n}\sum_{s=1}^{n}\sum_{\nu}\sum_{\nu'}\overset{\circ}{M}{}^{-1}_{is,\nu}\pi_{\nu}\,\overset{\circ}{M}{}^{-1}_{is,\nu'}\pi_{\nu'}$ we get the result.
Observe that the set of phase space coordinates $(\omega_{\mu}, L_{gv}; \overset{\circ}{\xi}_{i}, \pi_{i}\,;\, \beta_{\tau}, \mathcal{L}_{gv}, \phi_{gv})$ wouldn't be canonical.

II.5. The invariant volume element in new coordinates; basis

In this section we calculate the invariant volume element, i.v.e., on the $\mathbb{R}^{3n}$ in new coordinates $(\omega_{\mu}, \overset{\circ}{\xi}_{i}, \beta_{\tau})$. Note that the method in reference 24 determines the volume element only for the transversal, $\overset{\circ}{\xi}$ part. Authors claim that the remainder of the volume element is given by the i.v.e. of the group- or coset-manifolds. We determine them all explicitly. That comes from the fact that our Jacobi matrix $\overset{\circ}{M}_{\nu,is}$ does not equal π_{ν}^{is} from the reference 24 which we already mentioned in (II.2).

II.5.1 Proposition: The i.v.e. on the $\mathbb{R}^{3n}$ expressed in new coordinates $(\omega_{\mu}, \overset{\circ}{\xi}_{i}, \beta_{\tau})$ is

$$dm = (\overset{\circ}{\xi}_{1}\overset{\circ}{\xi}_{2}\overset{\circ}{\xi}_{3})^{n-3}\prod_{j<i}^{3}(\overset{\circ}{\xi}{}^{2}_{j} - \overset{\circ}{\xi}{}^{2}_{i})\prod_{i=1}^{3} d\overset{\circ}{\xi}_{i}\,\det {}^{R}\Psi_{\mu\mu'}(\omega)\prod_{\mu=1}^{3} d\omega_{\mu}\,\det {}^{L}\Psi_{\tau\tau'}(\beta)\prod_{\tau=1}^{3n-6} d$$

where $\det {}^{R}\Psi_{\mu\mu'}(\omega)\prod_{\mu} d\omega_{\mu}$ is the i.v.e. on the $SO(3)$ group manifold and $\det {}^{R}\Psi_{\tau\tau'}(\beta)\prod_{\tau} d\beta_{\tau}$ the i.v.e. on the right coset space $SO(n)/SO(n-3)$.

Proof: The i.v.e. on $\mathbb{R}^{3n}$ in new coordinates (ξ_{ν}) is given by the expression

$$dm = \Big[\det \sum_{l=1}^{3}\sum_{s=1}^{n} \overset{\circ}{M}_{\nu,ls}\,{}^{t}\overset{\circ}{M}_{ls,\nu'}\Big]^{\frac{1}{2}}\prod_{\nu=1}^{3n} d\xi_{\nu}$$

where $\mathcal{J}_{\nu,ls}$ is the Jacobi matrix for the transition $(\xi_{is}) \to (\xi_\nu)$. We insert for the set of coordinates $(\omega_\mu, \overset{\circ}{\xi}_i, \beta_\tau)$ matrix $\mathcal{J}$ given in (II.2.1), and after a long but straightforward calculation we get for expression $\sum_{l=1}^{3}\sum_{s=1}^{n} \mathcal{J}_{\nu,ls}\, {}^t\mathcal{J}_{ls,\nu'}$, from which the determinant should be calculated, the follwoing supermatrix form.

$$(\mathcal{J}\,{}^t\mathcal{J})_{\nu\nu'} = \begin{array}{c|ccc} & w' & \mu' & \tau' \\ \hline w & \delta_{ww'} & 0 & 0 \\ \mu & 0 & \gamma_{\mu\mu'} & \beta_{\mu\tau'} \\ \tau & 0 & \alpha_{\tau\mu'} & \delta_{\tau\tau'} \end{array}$$

where

$$\gamma_{\mu\mu'} := \sum_{i=1}^{3}\sum_{j=1}^{i-1} {}^R\Psi_{\mu,ji}(\omega)\,(\overset{\circ}{\xi}{}^2_j + \overset{\circ}{\xi}{}^2_i)\,{}^R\Psi_{\mu',ji}(\omega),$$

$$\beta_{\mu\tau'} := \sum_{i=1}^{3}\sum_{j=1}^{i-1} {}^R\Psi_{\mu,ji}(\omega)\,2\,\overset{\circ}{\xi}_j\,\overset{\circ}{\xi}_i\,{}^L\Psi_{\tau',ji}(\beta),$$

$$\alpha_{\tau\mu'} := \sum_{i=1}^{3}\sum_{j=1}^{i-1} {}^L\Psi_{\tau,ji}(\beta)\,2\,\overset{\circ}{\xi}_j\,\overset{\circ}{\xi}_i\,{}^R\Psi_{\mu',ji}(\omega),$$

$$\delta_{\tau\tau'} := \sum_{i=1}^{3}\sum_{j=1}^{i-1} {}^L\Psi_{\tau,ji}(\beta)\,(\overset{\circ}{\xi}{}^2_j + \overset{\circ}{\xi}{}^2_i)\,{}^L\Psi_{\tau',ji}(\beta) + \sum_{i=4}^{n}\sum_{j=1}^{3} {}^L\Psi_{\tau,ji}(\beta)\,\overset{\circ}{\xi}{}^2_j\,{}^L\Psi_{\tau',ji}(\beta).$$

The determinant of $(\mathcal{J}\,{}^t\mathcal{J})_{\nu\nu'}$ is given by

$$\det(\mathcal{J}\,{}^t\mathcal{J})_{\nu\nu'} = \det\gamma\,\det(\delta - \alpha\gamma^{-1}\beta) \qquad \text{(see ref. 45 p. 43).}$$

The submatrix γ we invert easily using (I.1.13). Further, using nothing but (I.1.13) we get the final result. That expressions $\det {}^L\Psi_{\mu\mu'}(\omega)\,d\omega_\mu$ and $\det {}^R\Psi_{\tau\tau'}(\beta)\,d\beta_\tau$ are i.v.e. of the $SO(3)$ group manifold and $SO(n)/SO(n-3)$ right coset manifold respectively, we conclude from (I.1.21) and (I.2.4). □

In order to complete this section we are going to write here

the basis elements of the Hilbert space determined in section 8 of reference 24 . Note once again that our point in the configuration space is given

$$\xi_{is} = \sum_{\ell=1}^{3} \sigma_{i\ell}(\omega)\ \overset{\circ}{\xi}_{\ell}\ \sigma'_{\ell s}(\beta)$$

where only $\overset{\circ}{\xi}_{\ell}$ are not some group parameters. Let us first have a look at a system of 3 relativ vectors

$$\xi_{is} = \sum_{\ell=1}^{3} (r_1)_{i\ell}\ \overset{\circ}{\xi}_{\ell}\ (r_2)_{\ell s}$$

where $r_1 \equiv \sigma(\omega)$ and $r_2 \in SO(3)$. If we want to write some basis functions in Hilbert space for this system for r_1 and r_2 parts we clearly take the rotation matrix elements $D^{L}_{KM}(r_1)$ and $D^{\mathcal{L}}_{\mathcal{K}\mathcal{M}}(r_2)$. L is the main quantum number of the angular momentum, K is its third projection "referred" to the body and M the third projection in laboratory frame. $\mathcal{L}$ is the main quantum number of the vortex spin , $\mathcal{K}$ is its third projection "in" the body and $\mathcal{M}$ the third projection in the 3-particle label frame. So one writes an ansatz for the functions on $g\mathcal{L}_+(3,\mathbb{R})$

$$\psi^{L\mathcal{L}}_{KM[\nu],\mathcal{K}\mathcal{M}}(\xi_{is}) \sim D^{L}_{KM}(r_1)\ D^{\mathcal{L}}_{\mathcal{K}\mathcal{M}}(r_2)\ f^{L\mathcal{L}}_{K[\nu]\mathcal{K}}(\overset{\circ}{\xi})$$

where $[\nu]$ denotes a set of linearly independent functions $f^{L\mathcal{L}}_{K[\nu]\mathcal{K}}(\overset{\circ}{\xi})$. The expression that we wrote is still an ansatz. We demand that $\psi^{L\mathcal{L}}_{KM[\nu],\mathcal{K}\mathcal{M}}(\xi_{is})$ should be invariant under the action of $diag(\mathcal{H}'_3 \times \mathcal{H}'_3)$ introduced in (II.1), which means simultaneous action of $\mathcal{H}'_3$ on both r_1 and r_2. Under this condition we get

$$\psi^{L\mathcal{L}}_{KM[\nu],\mathcal{K}\mathcal{M}}(\xi_{is}) = \left(1 + (-)^{K+\mathcal{K}}\right) f^{L\mathcal{L}}_{K[\nu]\mathcal{K}}(\overset{\circ}{\xi}) \left[D^{L}_{KM}(r_1) D^{\mathcal{L}}_{\mathcal{K}\mathcal{M}}(r_2) + (-)^{L-K+\mathcal{L}-\mathcal{K}} D^{L}_{-KM}(r_1) D^{\mathcal{L}}_{-\mathcal{K}\mathcal{M}}(r_2) \right.$$

Similar result have been already obtained in ref. 14 for the $S\mathcal{L}(3,\mathbb{R})$ scheme. The generalization for $n > 3$ starting from

$$\xi_{is} = \sum_{\ell=1}^{3} \sigma_{i\ell}(\omega)\ \overset{\circ}{\xi}_{\ell}\ \sigma'_{\ell s}(\beta)$$

is given by the straighforward extension of the $g\mathcal{L}_+(3,\mathbb{R})$ expression

$$\psi^{L\rho\mathcal{L}}_{KM[\nu]\delta\mathcal{K}\mathcal{M}} = \left(1 + (-)^{K+\mathcal{K}}\right) f^{L\rho\mathcal{L}}_{K[\nu]\delta}(\overset{\circ}{\xi}) \times$$

$$\times \left[D^{L}_{KM}(\omega)\ D^{\rho}_{\delta\mathcal{L}\mathcal{K}0,\mathcal{M}}(\beta) + (-)^{L-K+\mathcal{L}-\mathcal{K}} D^{L}_{-KM}(\omega)\ D^{\rho}_{\delta\mathcal{L}-\mathcal{K}0,\mathcal{M}}(\beta) \right]$$

where ρ denotes an irreducible unitary representation of SO(n) and $D^{\rho}_{\delta\mathcal{L}\mathcal{K}0,\mu}(\beta)$ we get out of $D^{\rho}_{\sigma,\mu}(\gamma)$ $[\sigma'(\gamma) \in SO(n)]$ where we take for the first index σ the subgroup chain

$$SO(n) > (SO(3) > SO(2)) \times (SO(n-3) > SO(n-4) \ldots) .$$

So instead of σ come $\delta\mathcal{L}\mathcal{K}\Omega$ where δ is a multiplicity index, $\mathcal{L}\mathcal{K}$ are the same as by $gl_{+}(3,R)$ and label the basis states $SO(3) > SO(2)$, and Ω labels the states for the SO(n-3) subgroup chain. In order to restrict $D(\gamma)$ to $D(\beta)$ where $\sigma'(\beta) \in SO(n) \mid SO(n-3)$ which is the right coset space, we set $\Omega=0$ which labels the identity representation of SO(n-3).

Now we should explain what the SO(n) group from the $SO(3) \times SO(n)$ coordinate scheme is good for. It can be shown[46] that the permutation group S_{n+1} can be embeded in O(n) in such a way that S_{n+1} through its representation $D^{[n,1]}(S_{n+1})$ is a subgroup of O_n and determines how n Jacobi vectors transform under the permutation of n+1 particles. From the fact that any transposition P_{st} s,t = 1,2,... n+1 has the determinant -1 and from the fact that $D^{[n+1]}$ is the representation of the S_n under which the center-of-mass transforms and clearly has the determinant equal +1, one concludes from the direct sum $D^{[n+1]}(P_{st}) + D^{[n,1]}(P_{st})$ that S_{n+1} via $D^{[n,1]}(S_{n+1})$ is not a subgroup of SO(n). But nevertheless $O(n) \cong Z_2 \times SO(n)$ where Z_2 is, as is well known, an abelian group whose representations are all one-dimensional. That is why D^{ρ} can be regarded also as an irreducible representation of O(n). So, for the second index μ of $D^{\rho}_{\sigma\mu}(\beta)$ we choose the group chain $O(n) > S_{n+1}$ and write $\mu \equiv \alpha\lambda\varkappa$ where α is a multiplicity index, λ labels an irrep of S_{n+1} and $\varkappa$ labels a basis for the representation. The final expression for the basis elements is

$$\psi^{L\rho\varkappa}_{KM[\nu]\delta\mathcal{K}\alpha\lambda\varkappa} = (1 + (-)^{K+\mathcal{K}}) \, f^{L\rho\mathcal{L}}_{K[\nu]\delta\mathcal{K}}(\xi) \times$$

$$\times \left[D^{L}_{KM}(\omega) \, D^{\rho}_{\delta\mathcal{L}\mathcal{K}0,\alpha\lambda\varkappa}(\beta) + (-)^{L-K+\mathcal{L}-\mathcal{K}} D^{L}_{-KM}(\omega) \, D^{\rho}_{\delta\mathcal{L}-\mathcal{K}0,\alpha\lambda\varkappa}(\beta) \right].$$

In the microscopic n+1 nuclear particle theory some concept for

the collective potential energy was needed, and was proposed by Vanagas [28] . He calls V_{coll} that part of the whole n+1 particle potential energy (given by 2 particle interaction) which is O(n) invariant. That brings us to the restriction by fixed collective energy to the Hilbert space functions with the fixed $D^{s}(\beta)$ part. The phenomenological collective potential energy of the form $V = V\left[tr\mathcal{T}, tr\mathcal{T}^2, tr\mathcal{T}^3 \right]$ is clearly also O(n) invariant because already $tr\mathcal{T}$, $tr\mathcal{T}^2$ and $tr\mathcal{T}^3$ are O(n) invariant.

II.6. The collective velocity field

In order to determine the linear velocity field in $SO(3)\times SO(n)$ scheme we first calculate π_{is} as a function of new coordinates and momenta $(\omega_\mu, L_{g\nu}; \overset{\circ}{\xi}_w \pi_w; \beta\tau; \mathcal{L}_{g\nu}, \phi_{g\nu})$

$$\pi_{is} = \sum_{\nu=1}^{3n} \overset{\omega}{\sigma}{}^{-1}_{is,\nu}\, \pi_\nu$$

$$= \sum_{j=1}^{3}\sum_{g=1}^{3}\sum_{\nu=g+1}^{3} \left(\sigma_{ig}(\omega)\sigma_{j\nu}(\omega) + \sigma_{i\nu}(\omega)\sigma_{jg}(\omega)\right) \frac{1}{\overset{\circ}{\xi}{}^2_\nu - \overset{\circ}{\xi}{}^2_g} L_{g\nu}\, \xi_{js}$$

$$+ \sum_{j=1}^{3}\sum_{w=1}^{3} \sigma_{iw}(\omega)\,\sigma_{jw}(\omega)\, \overset{\circ}{\xi}{}^{-1}_w\, \pi_w\, \xi_{js}$$

$$+ \sum_{j=1}^{3}\sum_{g=1}^{3}\sum_{\nu=g+1}^{3} \left(\sigma_{ig}(\omega)\,\frac{\overset{\circ}{\xi}_g}{\overset{\circ}{\xi}_\nu}\,\sigma_{j\nu}(\omega) + \sigma_{i\nu}(\omega)\,\frac{\overset{\circ}{\xi}_\nu}{\overset{\circ}{\xi}_g}\,\sigma_{jg}(\omega)\right) \frac{1}{\overset{\circ}{\xi}{}^2_\nu - \overset{\circ}{\xi}{}^2_g} \mathcal{L}_{g\nu}\, \xi_{js}$$

$$- \sum_{g=1}^{3}\sum_{\nu=4}^{n} \sigma_{ig}(\omega)\, \overset{\circ}{\xi}{}^{-1}_g\, \sigma'_{\nu s}(\beta)\, \phi_{g\nu} \qquad \text{(II.2.2)}$$

With respect to SO(n) (Vanagas') collectivity the first two lines in the expression for π_{is} determine the collective part. For lokal hamiltonians we have

$$m\vec{v}_s = \vec{\pi}_s$$

and as the authors in ref. 16 we extract the collective part of the velocity $\vec{v}_s$

$$v_{is}^{coll} = \frac{1}{m} \sum_{j=1}^{3} \left(\chi_{ij}^{vib} + \chi_{ij}^{s} \right) \xi_{js}$$

where

$$\chi_{ij}^{vib} := \sum_{w=1}^{3} \sigma_{iw}(\omega)\, \sigma_{jw}(\omega)\, \mathring{\xi}_w^{-1}\, \pi_w \,, \quad \chi_{ij}^{vib} = \chi_{ji}^{vib} \,;$$

$$\chi_{ij}^{s} := \sum_{g=1}^{3} \sum_{v=g+1}^{3} \left(\sigma_{ig}(\omega)\, \sigma_{jv}(\omega) + \sigma_{iv}(\omega)\, \sigma_{jg}(\omega) \right) \frac{1}{\mathring{\xi}_v^2 - \mathring{\xi}_g^2} L_{gv} \,, \quad \chi_{ij}^{s} = \chi_{ji}^{s} \,.$$

As the authors in ref. 16 we define the velocity field $\vec{v}$ at the point $\vec{\xi}$ to be the collective component of the velocity $\vec{v}_s$ for relative vector s at $\vec{\xi}_s \equiv \vec{\xi}$

$$v_i = \frac{1}{m} \sum_{j=1}^{3} \left[\chi_{ij}^{vib} + \chi_{ij}^{s} \right] \xi_j \,.$$

The velocity field is linear in the position vector $\vec{\xi}$. It is trivial to note that $\mathrm{rot}\,\vec{v} = 0$. So we conclude (as we also did concerning the kinetic energy) that the (collective) velocity field with respect to SO(n) collectivity is irrotational. It separates completely from the intrinsic part. But here we have the same complaint as by the kinetic energy: we are much more interested in some collective flow which would not be irrotational.

The first three lines of π_{is} are the candidates for the collective velocity w.r. t. $g\mathcal{L}_+(3,\mathbb{R})$ collectivity: the momenta given in (II.4.2) and (II.3.6) can still be seen as related to SO(3) submanifold of the right coset space $SO(n)/SO(n-3) \equiv C^R$ But the part proportional to ϕ's is related to the whole C^R and can not be defined as purely intrinsic w.r. t. $g\mathcal{L}_+(3,\mathbb{R})$ collectivity.

It should be stressed that all the classical analysis in this chapter was done on a phase space related to the special type of orbit in configuration space defined by the condition $\mathring{\xi}_1 > \mathring{\xi}_2 > \mathring{\xi}_3 > 0$. Whereas the other types of orbits in the quantum mechanical case are of measure zero (see (II.1) S. 43), in the classical case one should also consider all other types of orbits. Similar remarks apply also to the next chapter (see (III.1.3)).

III. The $gL_+(3,\mathbb{R})\times SO(n)$ scheme

III.0 Introduction

In this chapter we study the $gL_+(3,\mathbb{R})\times SO(n)$ scheme. In section III.1 we define the coordinate transformation (c.t.) based on the $gL_+(3,\mathbb{R})\times SO(n)$ action and corresponding orbit analysis. In this scheme collective coordinates are the group parameters which span the $gL_+(3,\mathbb{R})$ manifold, and the intrinsic coordinates are parameters of the right coset space SO(n)/SO(3)xSO(n-3). In section III.2 we determine the Jacobi matrix for the c.t. defined in (III.1) and invert it making use of certain nonintegrable constraints in analogy to some studies devoted to $gL_+(3,\mathbb{R})$ collectivity given by Gulshani and Rowe in ref. 16. By use of the Jacobi matrix so inverted we obtain the results for the kinetic energy and velocity field in new coordinates which are identical to ref. 16. By those expressions we find the collective and the intrinsic parts completely separated. In section III.3 we invert the Jacobi matrix exactly, without using any constraints and in (III.4) we define vector fields in new coordinates. The vortex spin $\tilde{\mathcal{L}}$ is now defined on an $SO(3)$ manifold explicitly and intrinsic differential operators $\tilde{\phi}$ on the right coset space SO(n)/SO(3)xSO(n-3). It should be stressed that $\tilde{\phi}$'s are not ϕ's from the $SO(3)\times SO(n)$ scheme just transformed, they are not any more orthogonal in the sense of ref. 25 to the vortex spin! In (III.5) we calculate the kinetic energy and try to construct the velocity field using the exact inverse Jacobi matrix. As expected from the fact that the vortex spin and the intrinsic differential operators are not any more orthogonal, coupling terms appear in the kinetic energy. The same holds true for the velocity field. We conclude that the separation into the intrinsic and collective parts can be obtained only when the coupling terms appearing in the kinetic energy and the velocity field become zero. It comes out of the analysis that they are zero when certain constraints are fulfilled which are exactly the constraints we propose (according to ref. 16) in (III.2) at the very beginning. But we do not see any reason from the point of view of Lie theory why this

constraint should be fulfilled and we conclude that the complete separation of the kinetic energy and velocity field into the collective and intrinsic parts w.r. t. $g\mathcal{L}_+(3,\mathbb{R})$ collectivity are not possible.

III.1. The $g\mathcal{L}_+(3,\mathbb{R})\times SO(n)$ scheme, orbit analysis

III.1.1 Definition: The action of $g\mathcal{L}_+(3,\mathbb{R})\times SO(n)$ group on $\mathbb{R}^{3n}$ given through the points which are the Cartesian coordinates of a system of n+1 particles described by n relative vectors $(\xi_{is}\,|\,i = 1,2,3$ $s = 1,2,\ldots,n)$ is introduced as the left action of $g\mathcal{L}_+(3,\mathbb{R})$ and the right action of SO(n):

$$\sum_{t=1}^{n}\sum_{\ell=1}^{3}(\tilde{\xi}')^{-1}_{i\ell}\,\xi_{\ell t}\,\tilde{\sigma}''_{ts}(\gamma) \qquad \text{where } \tilde{\xi}'\in g\mathcal{L}_+(3,\mathbb{R}),\ \tilde{\sigma}''(\gamma)\in SO(n).$$

We choose the representative point to be $\mathring{\xi}_{is}=\delta_{is}$ $i = 1,2,3$ $s = 1,2,\ldots,n$ and find its stability group by demanding that

$$\sum_{j=1}^{3}\sum_{t=1}^{n}(\tilde{\xi}')^{-1}_{i\ell}\,\delta_{\ell t}\,\tilde{\sigma}''_{ts}(\gamma) = \delta_{is}\,.$$

The stability group is isomorphic to SO(3)xSO(n-3), where SO(n-3) is a subgroup of SO(n) whereas SO(3) can be chosen to be either a subgroup of $g\mathcal{L}_+(3,\mathbb{R})$ or of SO(n). If we choose it to be the subgroup of $g\mathcal{L}_+(3,\mathbb{R})$ then we are again back to the SO(3)xSO(n) scheme which we already studied in chapter II. We choose SO(3) to be a subgroup of SO(n). Then the orbit for the representative point given above is the coset space $g\mathcal{L}_+(3,\mathbb{R})\times\left(SO(n)\,|\,SO(3)\times SO(n-3)\right)$ and is parametrized by 3n real parameters. The SO(n)/SO(3)xSO(n-3) is the right coset space. In order to show that the orbit discussed so far is the only one under the $g\mathcal{L}_+(3,\mathbb{R})\times SO(n)$ action, we now make use of the SO(3)xSO(n) scheme discussed in (II.1). From this scheme we know that $g\mathcal{L}_+(3,\mathbb{R})$ may be decomposed as

$$g\mathcal{L}_+(3,\mathbb{R}) = U\left\{\left(SO(3)\,|\,\mathcal{H}_3'\right)\times\mathring{\xi}\times SO(3)\right\}$$

We insert this decomposition of $g\mathcal{L}_+(3,\mathbb{R})$ into the coset space $g\mathcal{L}_+(3,\mathbb{R})\times\left(SO(n)\,|\,SO(3)\times SO(n-3)\right)$ and obtain

$\left(SO(3)/\mathcal{K}_3'\right) \times \mathring{\xi} \times SO(3) \times \left(SO(n)/SO(3)\times SO(n-3)\right)$ which is equivalent

to $\left(SO(3)/\mathcal{K}_3'\right) \times \mathring{\xi} \times \left(SO(n)/SO(n-3)\right)$. We are left with the

manifold decomposition as in SO(3)xSO(n) scheme which we had already proven to be complete (see (II.1)). So we conclude

III.1.2 Proposition: Under the action of $g\mathcal{L}_+(3,\mathbb{R}) \times SO(n)$, the $\mathbb{R}^{3n}$ obtains coordinates which are the parameters of the single orbit equivalent to the coset space $g\mathcal{L}_+(3,\mathbb{R}) \times \left(SO(n)/SO(3)\times SO(n-3)\right)$. We write for $\xi_{is} \in \mathbb{R}^{3n}$

$$\xi_{is} = \sum_{j=1}^{3} \xi'_{ij}\, \sigma''_{js}(\beta) \quad , \quad i = 1,2,3 \quad s = 1,2,\dots,n$$

where $\xi' \in g\mathcal{L}_+(3,\mathbb{R})$ and is parametrized by the real parameters $g\mathcal{L}_+(3,\mathbb{R})$ matrix elements $\left(\xi'_{ij} \mid i,j = 1,2,3\right)$ and $\sigma''(\beta) \in SO(n)/SO(3)\times SO(n-$ is the right coset space. This decomposition is unique. Coordinates ξ'_{ij} can be considered as 9 collective coordinates in the sense of $g\mathcal{L}_+(3,\mathbb{R})$ collectivity and β's are then 3n-9 intrinsic coordinates. We define this c.t. as Φ_1

$$\left(\xi_{is} \mid i = 1,2,3 \;\; s = 1,2,\dots,n\right) \xrightarrow{\Phi_1} \left(\xi'_{ie}, \beta_\tau \mid i,e = 1,2,3 \;\; \tau = 1,2,\dots,3n-9\right)$$

$$\equiv \left(\zeta_\nu \mid \nu = 1,2,\dots,3n\right). \quad \square$$

The c.t. Φ_1 is not that physical: we would like to obtain some collective coordinates which we can interpret as Euler angles, coordinates for vibrations and deformations and coordinates for vorticities. These coordinates can be easily introduced using $SO(3)\times SO(m)$ scheme for n = 3 already mentioned above, which provide us with the well known decomposition for the $g\mathcal{L}_+(3,\mathbb{R})$. So we introduce c.t. Φ_2 and Φ which is defined as $\Phi := \Phi_2 \circ \Phi_1$.

III.1.3 Proposition: The coordinate transformation

$$\left(\xi_{is} \mid i = 1,2,3 \;\; s = 1,2,\dots,n\right) \xrightarrow{\Phi} \left(\omega_\mu, \mathring{\xi}_i, \omega'_{\mu'}, \beta_\tau \mid_{\mu,\mu',i=1,2,3\;\tau=1,2,\dots,3n-9}\right)$$

is defined by the unique decomposition of ξ_{is}

$$\xi_{is} = \sum_{\ell=1}^{3} \sum_{k=1}^{3} \sigma_{i\ell}(\omega)\, \mathring{\xi}_\ell\, \sigma'_{\ell k}(\omega')\, \sigma''_{ks}(\beta) \qquad \text{where } \sigma(\omega) \in SO(3)/\mathcal{K}'_3$$

(for $\mathcal{K}'_3$ see (II.1), p. 41); $\mathring{\xi}_1 > \mathring{\xi}_2 > \mathring{\xi}_3 > 0$, $\mathring{\xi}_i \in \mathbb{R}^1$; $\sigma'(\omega') \in SO(3)$; $\sigma''(\beta) \in SO(n)/SO(3) \times SO(n-3)$. This c.t. Φ we introduce by two successive c.t. $\Phi_2 \circ \Phi_1$ where Φ_1 is as in (III.1.2) and Φ_2 is

$$(\xi'_{j\ell}, \beta_\tau / j,\ell = 1,2,3\ \tau = 1,2,\dots,3n-9) \xrightarrow{\Phi_2} (\omega_\mu, \mathring{\xi}_i, \omega'_{\mu'}, \beta_\tau / \mu,\mu', i = 1,2,3\ \tau = 1,2,\dots,3n-9)$$

introduced through the unique decomposition of ξ'

$$\xi'_{ik} = \sum_{\ell=1}^{3} \sigma_{i\ell}(\omega)\, \mathring{\xi}_\ell\, \sigma'_{\ell k}(\omega'). \qquad \square$$

III.2. Jacobi matrix for the coordinate transformation Φ, inverse Jacobi matrix obtained using the constraints, kinetic energy, velocity field

In this section using our methods we reconstruct the results from the ref. 16 . In this reference authors Gulshani and Rowe obtained complete separation of the kinetic energy and velocity field w.r. t. $\mathcal{GL}_+(3,\mathbb{R})$ collectivity using from the very beginning certain nonholonomic constraints introduced ad hoc, to reduce redundant coordinates. We study here the c.t. Φ introduced in (III.1.3).

III.2.1 Proposition: The Jacobi matrix for the c.t. Φ introduced in (III.1.3) is given by its matrix elements

$$\frac{\partial \xi_{is}}{\partial \omega_\mu} = \sum_{g=1}^{3} \sum_{j=1}^{3} \sum_{\ell=1}^{3} \sigma_{ij}(\omega)\, {}^{R}\widetilde{\Psi}_{\mu,j\ell}(\omega)\, \mathring{\xi}_\ell\, \sigma'_{\ell g}(\omega')\, \sigma''_{gs}(\beta)$$

$$\frac{\partial \xi_{is}}{\partial \mathring{\xi}_w} = \sum_{g=1}^{3} \sigma_{iw}(\omega)\, \sigma'_{wg}(\omega')\, \sigma''_{gs}(\beta)$$

$$\frac{\partial \xi_{is}}{\partial \omega'_\mu} = \sum_{\ell=1}^{3} \sum_{j=1}^{3} \sum_{g=1}^{3} \sigma_{i\ell}(\omega)\, \mathring{\xi}_\ell\, {}^{L}\widetilde{\Psi}_{\mu,\ell j}(\omega')\, \sigma'_{jg}(\omega')\, \sigma''_{gs}(\beta)$$

$$\frac{\partial \xi_{is}}{\partial \beta_{\tau t}} = \sum_{\ell=1}^{3} \sum_{q=1}^{3} \sum_{t=1}^{n} \sigma_{i\ell}(\omega) \overset{\circ}{\xi}_{\ell} \sigma'_{\ell q}(\omega') \, {}^{L}\widetilde{\mathcal{F}}_{\tau, qt}(\beta) \, \sigma''_{ts}(\beta) \,. \quad \square$$

III.2.2 Proposition: The inverse Jacobi matrix $\mathcal{J}^{-1}$ can be found with the ansatz

$$\frac{\partial \omega_{\mu}}{\partial \xi_{is}} = \sum_{q=1}^{3} \sum_{v=q}^{3} \sum_{b=1}^{3} \sigma_{iq}(\omega) \frac{1}{2} \left[{}^{R}\Theta_{qv,\mu}(\omega) + A_{qv,\mu} \right] \overset{\circ}{\xi}_{v}^{-1} \sigma'_{vb}(\omega') \, \sigma''_{bs}(\beta)$$

$$- \sum_{v=1}^{3} \sum_{q=v+1}^{3} \sum_{b=1}^{3} \sigma_{iq}(\omega) \frac{1}{2} \left[{}^{R}\Theta_{vq,\mu}(\omega) - A_{vq,\mu} \right] \overset{\circ}{\xi}_{v}^{-1} \sigma'_{vb}(\omega') \, \sigma''_{bs}(\beta)$$

$$\frac{\partial \overset{\circ}{\xi}_{w}}{\partial \xi_{is}} = \sum_{b=1}^{3} \sigma_{iw}(\omega) \, \sigma'_{wb}(\omega') \, \sigma''_{bs}(\beta)$$

$$\frac{\partial \omega'_{\mu}}{\partial \xi_{is}} = \sum_{q=1}^{3} \sum_{v=q}^{3} \sum_{b=1}^{3} \sigma_{iq}(\omega) \overset{\circ}{\xi}_{q}^{-1} \frac{1}{2} \left[{}^{L}\Theta_{qv,\mu}(\omega') + B'_{qv,\mu} \right] \sigma'_{vb}(\omega') \, \sigma''_{bs}(\beta)$$

$$- \sum_{v=1}^{3} \sum_{q=v+1}^{3} \sum_{b=1}^{3} \sigma_{iq}(\omega) \overset{\circ}{\xi}_{q}^{-1} \frac{1}{2} \left[{}^{L}\Theta_{vq,\mu}(\omega') - B'_{vq,\mu} \right] \sigma'_{vb}(\omega') \, \sigma''_{bs}(\beta)$$

$$\frac{\partial \beta_{\tau t}}{\partial \xi_{is}} = \sum_{b=1}^{3} \sum_{v=4}^{n} \sum_{q=1}^{3} \sigma_{iq}(\omega) \overset{\circ}{\xi}_{q}^{-1} \sigma'_{qb}(\omega') \frac{1}{2} \left[{}^{L}\Theta_{bv,\tau}(\beta) + B_{bv,\tau} \right] \sigma''_{vs}(\beta)$$

and with the constraints ${}^{L}\widetilde{\mathcal{F}}_{\tau, bv}(\beta) \equiv 0$ for $b < v = 1, 2, 3$.
The inverse Jacobi matrix $\mathcal{J}^{-1}$ is given by its matrix elements

$$\frac{\partial \omega_{\mu}}{\partial \xi_{is}} = \sum_{\substack{v=1 \\ (v \neq q)}}^{3} \sum_{q=1}^{3} \sum_{b=1}^{3} \sigma_{iq}(\omega) \frac{\overset{\circ}{\xi}_{v}}{\overset{\circ}{\xi}_{v} - \overset{\circ}{\xi}_{q}} \, {}^{R}\widetilde{\Theta}_{qv,\mu}(\omega) \, \sigma'_{vb}(\omega') \, \sigma''_{bs}(\beta)$$

$$\frac{\partial \overset{\circ}{\xi}_{w}}{\partial \xi_{is}} = \sum_{b=1}^{3} \sigma_{iw}(\omega) \, \sigma'_{wb}(\omega') \, \sigma''_{bs}(\beta)$$

$$\frac{\partial \omega'_\mu}{\partial \xi_{is}} = \sum_{\substack{v=1 \\ (v\neq g)}}^{3} \sum_{g=1}^{3} \sum_{b=1}^{3} \sigma_{ig}(\omega) \frac{\overset{\circ}{\xi}_g}{\overset{\circ}{\xi}{}^2_g - \overset{\circ}{\xi}{}^2_v} \; {}^{L}\widetilde{\Theta}_{gv,\mu}(\omega') \, \sigma'_{vb}(\omega') \, \sigma''_{bs}(\beta)$$

$$\frac{\partial \beta_\tau}{\partial \xi_{is}} = \sum_{v=4}^{n} \sum_{g=1}^{3} \sum_{b=1}^{3} \sigma_{ig}(\omega) \, \overset{\circ}{\xi}{}^{-1}_g \, \sigma'_{gb}(\omega') \; {}^{L}\Theta_{bv,\tau'}(\beta) \, \sigma''_{vs}(\beta) . \quad \square$$

It should be noted that there is no reason to demand that ${}^{L}\Psi_{\tau,bv}(\beta) \equiv 0$ for $b<v=1,2,3$. Those are the pure constraints which we need in order to be able to invert the Jacobi matrix with the ansatz given in (III.2.2). Those constraints correspond to the ones given in ref. 16 by relation (2.9)

$$\sum_{s=1}^{n} \left(\sigma''_{bs}(\beta) \frac{\partial \sigma''_{vs}(\beta)}{\partial \beta_\tau} - \sigma''_{vs}(\beta) \frac{\partial \sigma''_{bs}(\beta)}{\partial \beta_\tau} \right) = 2 \, {}^{L}\widetilde{\Psi}_{\tau,bv} \quad b,v=1,2,3 \text{ (see (I.4.2))}$$
$$= 0 .$$

Actually in ref. 16 σ'' is some 3xn matrix which is not identified as $\sigma'' \in SO(n)/SO(3)xSO(n-3)$, and the relation (2.9) means 3 from 9 constraints on σ''_{bs} there denoted by $x''_{n\alpha}$ $n = 1,2,\ldots,N$ $(N = A-1)$ and $\alpha = 1,2,3$. The 6 other holonomic constraints are such that one can identify x'' by σ'' . The constraints in (2.9) are nonholonomic. Once again: we do not need them because we have no superfluous coordinates, c.t. Φ provides us with exactly 3n new coordinates. Having the expression for the inverse Jacobi matrix, we easily calculate the expression for the kinetic energy in new coordinates.

III.2.3 Proposition: The quantum mechanical kinetic energy on $\mathbb{R}^{3n}$ is given in the new set of coordinates $(\omega_\mu, \overset{\circ}{\xi}_\ell, \omega'_{\mu'}, \beta_\tau)$ by

$$T = \frac{1}{2m} \sum_{g=1}^{3} \sum_{v=g+1}^{3} \frac{\overset{\circ}{\xi}{}^2_v + \overset{\circ}{\xi}{}^2_g}{\left(\overset{\circ}{\xi}{}^2_v - \overset{\circ}{\xi}{}^2_g\right)^2} \; \overset{\wedge}{\mathcal{L}}{}^2_{gv}$$

$$+ \frac{1}{2m} \sum_{g=1}^{3} \sum_{v=g+1}^{3} \frac{\overset{\circ}{\xi}{}^2_v + \overset{\circ}{\xi}{}^2_g}{\left(\overset{\circ}{\xi}{}^2_v - \overset{\circ}{\xi}{}^2_g\right)^2} \; \overset{\wedge}{\widetilde{\mathcal{L}}}{}^2_{gv}$$

$$+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{4\,\mathring{\xi}_v\mathring{\xi}_g}{(\mathring{\xi}_v^2-\mathring{\xi}_g^2)^2}\,\hat{L}_{gv}\hat{\tilde{\mathcal{L}}}_{gv}$$

$$+\frac{1}{2m}\sum_{v=1}^{3}\hat{t}_v^2$$

$$-\frac{ih}{2m}\left\{\sum_{v=1}^{3}\frac{n-3}{\mathring{\xi}_v}\,\hat{t}_v+\sum_{v=1}^{3}\sum_{\substack{g=1\\(v\neq g)}}^{3}\frac{2\,\mathring{\xi}_v}{\mathring{\xi}_v^2-\mathring{\xi}_g^2}\,\hat{t}_v\right\}$$

$$+\frac{1}{2m}\sum_{g=1}^{3}\sum_{v=4}^{n}\mathring{\xi}_g^{-2}\,\hat{\tilde{\phi}}_{gv}^2$$

where $\hat{L}_{gv}:=-ih\sum_{\mu=1}^{3}{}^{R}\Theta_{gv,\mu}(\omega)\frac{\partial}{\partial\omega_\mu}$ $g<v=1,2,3$ is the angular momentum referred to the body; $\hat{\tilde{\mathcal{L}}}_{gv}:=ih\sum_{\mu=1}^{3}{}^{L}\Theta_{gv,\mu}(\omega')\frac{\partial}{\partial\omega'_\mu}$ $g<v=1,2,3$ is vortex spin defined on an $SO(3)$ manifold; $\hat{t}_v:=-ih\frac{\partial}{\partial\mathring{\xi}_v}$ $v=1,2,3$ are vibrational momenta and

$$\hat{\tilde{\phi}}_{gv}:=ih\sum_{\tau=1}^{3n-9}\sum_{b=1}^{3}\sigma'_{gb}(\omega')\,{}^{L}\Theta_{bv,\tau}(\beta)\frac{\partial}{\partial\beta_\tau}\qquad g=1,2,3\quad v=4,5,\dots,n$$

are intrinsic momenta. □

This expression for the kinetic energy in new coordinates can be interpreted as the complete separation of the kinetic energy into the collective and intrinsic parts, of course only assuming that the constraints postulated in (III.2.2) makes any sense. Now we try to determine the (collective) velocity field. Using the result of (III.2.2) we can relate the old Cartesian momenta π_{is} with the new momenta π_ν canonically conjugate to the new coordinates given by ϕ

$$\pi_{is}=\sum_{\nu=1}^{3n}M^{-1}_{is,\nu}\,\pi_\nu$$

$$=\sum_{j=1}^{3}\Big[\sum_{w=1}^{3}\sigma_{iw}(\omega)\,\sigma_{jw}(\omega)\,\mathring{\xi}_w^{-1}\,\pi_w$$

$$+\sum_{g=1}^{3}\sum_{v=g+1}^{3}\left(\sigma_{ig}(\omega)\,\sigma_{jv}(\omega)+\sigma_{iv}(\omega)\,\sigma_{jg}(\omega)\right)\frac{1}{\overset{\circ}{\xi}{}^{2}_{v}-\overset{\circ}{\xi}{}^{2}_{g}}\,L_{gv}$$

$$+\frac{1}{2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\left(\sigma_{jg}(\omega)\,\sigma_{iv}(\omega)+\sigma_{iv}(\omega)\,\sigma_{jv}(\omega)\right)\frac{1}{\overset{\circ}{\xi}_{g}\,\overset{\circ}{\xi}_{v}}\,\frac{\overset{\circ}{\xi}{}^{2}_{v}+\overset{\circ}{\xi}{}^{2}_{g}}{\overset{\circ}{\xi}{}^{2}_{v}-\overset{\circ}{\xi}{}^{2}_{g}}\,\tilde{\mathcal{L}}_{gv}$$

$$+\frac{1}{2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\left(\sigma_{jg}(\omega)\,\sigma_{iv}(\omega)-\sigma_{ig}(\omega)\,\sigma_{jv}(\omega)\right)\frac{1}{\overset{\circ}{\xi}_{g}\,\overset{\circ}{\xi}_{v}}\,\tilde{\mathcal{L}}_{gv}\Big]\,\xi_{js}$$

$$+\sum_{\alpha=1}^{3n-9}\frac{\partial\beta_{\alpha}}{\partial\,\xi_{is}}\,\pi_{\alpha}\,.$$

In the last expression we partially introduced some moments which are not any more canonically conjugate to the coordinates given by Φ as the classical total angular momenta referred to the body

$$L_{gv}:=\sum_{\mu=1}^{3}{}^{R}\Theta_{gv,\mu}(\omega)\,\pi_{\mu}\qquad g<v=1,2,3$$

and the classical vortex spin

$$\tilde{\mathcal{L}}_{gv}:=-\sum_{\mu'=1}^{3}{}^{L}\Theta_{gv,\mu'}(\omega')\,\pi_{\mu'}\qquad g<v=1,2,3\,.$$

The velocity corresponding to the relative vector s can be defined as

$$m\vec{v}_{s}=\vec{\pi}_{s}$$

and the authors in ref. 16 extract v_{is}^{coll} as

$$v_{is}^{coll}=\frac{1}{m}\sum_{j=1}^{3}\left(\chi_{ij}^{vib}+\chi_{ij}^{S}+\chi_{ij}^{A}\right)\xi_{js}$$

where

$$\chi_{ij}^{vib}:=\sum_{w=1}^{3}\sigma_{iw}(\omega)\,\sigma_{jw}(\omega)\,\overset{\circ}{\xi}{}^{-1}_{w}\,\pi_{w}\,,\qquad \chi_{ij}^{vib}=\chi_{ji}^{vib}\,;$$

$$\chi_{ij}^{S}:=\frac{1}{2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\left(\sigma_{jg}(\omega)\,\sigma_{iv}(\omega)+\sigma_{ig}(\omega)\,\sigma_{jv}(\omega)\right)\frac{1}{\overset{\circ}{\xi}_{g}\,\overset{\circ}{\xi}_{v}}\,\frac{\overset{\circ}{\xi}{}^{2}_{v}+\overset{\circ}{\xi}{}^{2}_{g}}{\overset{\circ}{\xi}{}^{2}_{v}-\overset{\circ}{\xi}{}^{2}_{g}}\,\tilde{\mathcal{L}}_{gv}$$

$$+ \sum_{g=1}^{3} \sum_{v=g+1}^{3} \left(\sigma_{ig}(\omega)\sigma_{jv}(\omega) + \sigma_{iv}(\omega)\sigma_{jg}(\omega)\right) \frac{1}{\mathring{\xi}_v^2 - \mathring{\xi}_g^2} L_{gv}^2 , \quad \chi^S_{ij} = \chi^S_{ji} ;$$

$$\chi^A_{ij} := \frac{1}{2} \sum_{g=1}^{3} \sum_{v=g+1}^{3} \left(\sigma_{jg}(\omega)\sigma_{iv}(\omega) - \sigma_{ig}(\omega)\sigma_{jv}(\omega)\right) \frac{1}{\mathring{\xi}_g \mathring{\xi}_v} \tilde{\mathcal{L}}_{gv} , \quad \chi^A_{ij} = -\chi^A_{ji} .$$

It is clear that χ^{vib}, χ^S and χ^A are pure collective. The authors in 16 define further velocity field $\vec{v}$ at the point $\vec{\xi}$ to be the collective component of the velocity $\vec{v}_s$ for relative vectors at $\vec{\xi}_s = \vec{\xi}$

$$v_i = \frac{1}{m} \sum_{j=1}^{3} \left[\chi^{vib}_{ij} + \chi^S_{ij} + \chi^A_{ij} \right] \xi_j .$$

The velocity field is linear in the position vector $\vec{\xi}$.
If one demands that the velocity field be irrotational

$$\mathrm{rot}\, \vec{v} = 0$$

one gets the condition

$$\chi^A_{ij} = 0$$

or equivalently,

$$\tilde{\mathcal{L}}_{gv} = 0 \quad \text{for} \quad g < v = 1,2,3.$$

The expression for the classical kinetic energy can be obtained in the same way as in (II.4.2), using (III.2.2). The result is exactly the same as in (III.2.3), only that instead of differential operators there are written the corresponding classical momenta. If we put in the expression for the collective classical kinetic energy, the condition $\tilde{\mathcal{L}}_{gv} = 0$, we obtain the irrotational-flow expression

$$T_{rot}\left(\tilde{\mathcal{L}} = 0\right) = \frac{1}{2m} \sum_{g=1}^{3} \sum_{v=g+1}^{3} \frac{\mathring{\xi}_v^2 + \mathring{\xi}_g^2}{\left(\mathring{\xi}_v^2 - \mathring{\xi}_g^2\right)^2} L_{gv}^2$$

for the rotational kinetic energy. If we put this expression in the form

$$T_{rot}(\tilde{\Sigma}=0) = \frac{1}{2}\sum_{g=1}^{3}\sum_{v=g+1}^{3}\frac{1}{(I_{gv})_{irr}}L_{gv}^{2}$$

and remember that L_{gv} is the angular momentum referred to the body frame, we can read off the effective principal moments of inertia for irrotational flow

$$(I_{gv})_{irr} = \frac{m(\overset{\circ}{\xi}{}_{v}^{2} - \overset{\circ}{\xi}{}_{g}^{2})^{2}}{\overset{\circ}{\xi}{}_{v}^{2} + \overset{\circ}{\xi}{}_{g}^{2}} .$$

In the ref. 41 in phase space under the condition that the collective hamiltonian H_{coll} is O(n) invariant, among all possible orbits (see ref. 30 and 47) under the action of $\mathfrak{Sp}(6, \mathbb{R})\times O(n, \mathbb{R})$, a degenerate orbit has been found. This orbit is the coset space $\mathfrak{Sp}(6, \mathbb{R})\times O(n, \mathbb{R})/T(3(3+1)/2)\wedge \mathrm{diag}(O(3, \mathbb{R})\times O(3, \mathbb{R}))\times O(n-3, \mathbb{R})$ and has the dimension 3n+6. On this orbit the velocity field takes the form

$$v_i = \frac{1}{m}\sum_{j=1}^{3} Z_{ij}\xi_j$$

where $Z_{ij} = Z_{ji}$, and $\vec{v}$ is clearly irrotational. It can be also shown that on this orbit all the differential operators defined on the O(n) manifold are identically zero and among them is the vortex spin. So, irrotational flow comes out as a possible concept. It should be stressed that the analysis of vortex-free fluidity in ref. 41 and also in 16 are purely classical. Existence of an vortex-free fluid quantum mechanically, or even in some intermediate region between classical physics and quantum mechanics, would be questionable.

The limes where the rigid movement of the system is expected has been particularly clearly discussed in ref. 23 and is stated as the case when all "internal" velocities (velocities in the body) vanish. In our calculation we did not introduce concepts of velocities but rather of the momenta.

A very successful description of the collective rotations quantum mechanically making use of the geometric, coordinate description has been done in the ref. 48 . The authors introduced an internal frame which differs from the body fixed frame via

a rotation which depends only on internal coordinates. The rotation is not fixed so that the definition of the internal frame becomes the variational freedom which is used in order to decouple the intrinsic motion from the collective rotation in the best possible way.

III.3. The Jacobi matrix and its inverse for the coordinate transformations Φ_1, Φ_2 and Φ

In this section we are not going to find the explicit inverse Jacobi matrix for the coordinate transformation Φ. We will determine the Jacobi matrices and their inverse for Φ_1 and Φ_2. In all concrete calculations in (III.4) and (III.5) we will proceed in two steps: first find expressions given after the coordinate transformation Φ_1 using $\mathcal{J}^{-1}$ for Φ_1 and then apply Φ_2 using $\mathcal{J}^{-1}$ for Φ_2 in order to determine the final form of expressions studied in the coordinates related to Φ.

III.3.1 Proposition: The Jacobi matrix for the coordinate transformation Φ_1 introduced in (III.1.2)

$$(\xi_{is} \mid i=1,2,3 \quad s=1,2,\dots,n) \xrightarrow{\Phi_1} (\xi'_{il}, \beta_\alpha \mid i,l=1,2,3 \quad \alpha=1,2,\dots,3n-9) \equiv (\xi_\nu \mid \nu=1,2,\dots,3n)$$

is given through its matrix elements

$$\mathcal{J}_{\nu,is} \longrightarrow \begin{bmatrix} \mathcal{J}_{jl,is} \\ \mathcal{J}_{\alpha,is} \end{bmatrix} \equiv \begin{bmatrix} \frac{\partial \xi_{is}}{\partial \xi'_{jl}} \\ \frac{\partial \xi_{is}}{\partial \beta_\alpha} \end{bmatrix}$$

(I) $$\frac{\partial \xi_{is}}{\partial \xi'_{jl}} = \delta_{ij}\, \sigma''_{ls}(\beta),$$

(II) $$\frac{\partial \xi_{is}}{\partial \beta_\alpha} = \sum_{m=1}^{3} \sum_{k=1}^{n} \tilde{\Psi}_{\alpha,mk}(\beta)\, \xi'_{im}\, \sigma''_{ks}(\beta).$$

Proof: It follows easily from (I.4.1). □

III.3.2 Proposition: The inverse Jacobi matrix $\mathcal{J}^{-1}$ for the coordinate transformation Φ_1 introduced in (III.1.2) is given by its matrix elements

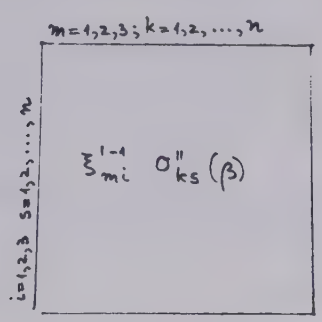
$$\mathcal{J}^{-1} \to \boxed{\mathcal{J}^{-1}_{is,jl} \quad \mathcal{J}^{-1}_{is,\tau}} \equiv \boxed{\frac{\partial \xi'_{jl}}{\partial \xi_{is}} \quad \frac{\partial \beta_\tau}{\partial \xi_{is}}}$$

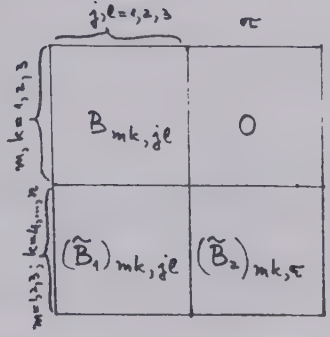
$$\text{(I)} \quad \frac{\partial \xi'_{jl}}{\partial \xi_{is}} = \delta_{ij}\, \sigma''_{ls}(\beta) + \sum_{m=1}^{3} \sum_{k=1}^{3} \sum_{m'=1}^{3} \sum_{k'=4}^{n} \xi'^{-1}_{m'i}\, \sigma''_{k's}(\beta)\, {}^{L}\Theta_{m'k',mk}(\beta)\, \xi'_{jm}\, \delta_{kl}$$

$$\text{(II)} \quad \frac{\partial \beta_\tau}{\partial \xi_{is}} = \sum_{m=1}^{3} \sum_{k=4}^{n} \xi'^{-1}_{mi}\, \sigma''_{ks}(\beta)\, {}^{L}\Theta_{mk,\tau}(\beta).$$

Proof: One makes the following ansatz

$$\mathcal{J}^{-1}_{\{is,\ \alpha\}\ \{jl,\ \tau\}} = \sum_{m=1}^{3} \sum_{k=1}^{n} \xi'^{-1}_{mi}\, \sigma''_{ks}(\beta)$$





and using (I.2.3) we conclude that

$$B_{mk,jl} = \xi'_{jm}\, \delta_{kl}\,, \quad (\tilde{B}_1)_{mk,jl} = \sum_{m'=1}^{3} \sum_{k'=1}^{3} {}^{L}\Theta_{mk,m'k'}(\beta)\, \xi'_{jm'}\, \delta_{k'l}$$

and $(\tilde{B}_2)_{mk,\tau} = {}^{L}\Theta_{mk,\tau}(\beta)$. □

III.3.3 Proposition: The Jacobi matrix for the c.t. Φ_2

$$(\xi'_{jl}, \beta_\tau \,|\, j,l=1,2,3\ \tau=1,2,\ldots,3n-9) \xrightarrow{\Phi_2} (\omega_\mu, \mathring{\xi}_i, \omega'_{\mu'}, \beta_\tau \,|\, \mu,\mu', i=1,2,3\ \tau=1,2,\ldots 3n-9)$$

introduced in (III.1.3) is given through its nontrivial matrix elements identical to those obtained for $SO(3) \times SO(n)$ scheme in

(II.2.1) setting $n = 3$, instead of ξ_{ls} $l = 1,2,3$ $s = 1,2,..,n$, ξ'_{jl} $j,l = 1,2,3$ and instead of β_π, $\omega'_{\mu'}$

$$\text{(I)} \quad \frac{\partial \xi'_{jl}}{\partial \omega_\mu} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sigma_{jk}(\omega)\, {}^R\tilde{\mathcal{F}}_{\mu,ki}(\omega)\, \overset{\circ}{\xi}_i\, \sigma'_{il}(\omega')$$

$$\text{(II)} \quad \frac{\partial \xi'_{jl}}{\partial \overset{\circ}{\xi}_w} = \sigma_{jw}(\omega)\, \sigma'_{wl}(\omega')$$

$$\text{(III)} \quad \frac{\partial \xi'_{jl}}{\partial \omega'_{\mu'}} = \sum_{i=1}^{3} \sum_{k=1}^{3} \sigma_{ji}(\omega)\, \overset{\circ}{\xi}_i\, {}^L\tilde{\mathcal{F}}_{\mu',ik}(\omega')\, \sigma'_{kl}(\omega'). \quad \square$$

III.3.4 Proposition: The inverse Jacobi matrix $\mathcal{J}^{-1}$ for the c.t. Φ_2 introduced in (III.1.3) is given through its nontrivial matrix elements identical to those obtained for $SO(3)\times SO(n)$ scheme in (II.2.2) setting $n = 3$, instead of ξ_{ls} $l = 1,2,3$ $s = 1,2,..,n$ ξ'_{jl} $j,l = 1,2,3$, and instead of β_π, $\omega'_{\mu'}$.

$$\text{(I)} \quad \frac{\partial \omega_\mu}{\partial \xi'_{jl}} = \sum_{\nu=1}^{3} \sum_{g=1}^{3} \sigma_{jg}(\omega)\, \frac{\overset{\circ}{\xi}_\nu}{\overset{\circ}{\xi}{}^2_\nu - \overset{\circ}{\xi}{}^2_g}\, {}^R\tilde{\Theta}_{g\nu,\mu}(\omega)\, \sigma'_{\nu l}(\omega')$$

$$\text{(II)} \quad \frac{\partial \overset{\circ}{\xi}_w}{\partial \xi'_{jl}} = \sigma_{jw}(\omega)\, \sigma'_{wl}(\omega')$$

$$\text{(III)} \quad \frac{\partial \omega'_{\mu'}}{\partial \xi'_{jl}} = \sum_{\nu=1}^{3} \sum_{g=1}^{3} \sigma_{jg}(\omega)\, \frac{\overset{\circ}{\xi}_g}{\overset{\circ}{\xi}{}^2_g - \overset{\circ}{\xi}{}^2_\nu}\, {}^L\tilde{\Theta}_{g\nu,\mu'}(\omega')\, \sigma'_{\nu l}(\omega'). \quad \square$$

III.4. The left and the right action generators and other differential operators expressed through the new coordinates after the coordinate transformation Φ_1 and Φ_2

In this section we consider only differential operators and we claim that the classical momenta can be obtained analogously.

III.4.1 Proposition: The right action generaters of SO(n) acting on the manifold $gL_+(3,\mathbb{R})\times\big(SO(n)\,|\,SO(3)\times SO(n-3)\big)$, expressed through the new ξ' and β coordinates after the c.t. Φ_1 given in (III.1.2) are

$$ {}^{R}X^{a}_{st} = \sum_{m=1}^{3}\sum_{\ell=m+1}^{3} {}^{R}\Theta_{st,m\ell}(\beta)\sum_{j=1}^{3}\left(\xi'_{jm}\frac{\partial}{\partial\xi'_{j\ell}} - \xi'_{j\ell}\frac{\partial}{\partial\xi'_{jm}}\right) + {}^{R}L_{st}\,, $$

$s<t=1,2,\dots,n$ where

$$ {}^{R}L_{st} := \sum_{\sigma=1}^{3n-9} {}^{R}\Theta_{st,\sigma}(\beta)\frac{\partial}{\partial\beta_\sigma}\,. $$

After the successive Φ_2 transformation, ${}^{R}X^{a}_{st}$ takes the form

$$ {}^{R}X^{a}_{st} = -\sum_{m=1}^{3}\sum_{\ell=m+1}^{3}\sum_{q=1}^{3}\sum_{v=q+1}^{3} {}^{R}\Theta_{st,m\ell}(\beta)\,{}^{t}d_{m\ell,qv}(\omega')\,\tilde{\mathcal{L}}_{qv} + {}^{R}L_{st} $$

where $\tilde{\mathcal{L}}_{qv} := -\sum_{\mu'=1}^{3} {}^{L}\Theta_{qv,\mu'}(\omega')\frac{\partial}{\partial\omega'_{\mu'}}$ $\quad q<v=1,2,3$ is the vortex spin defined on the $SO(3)$ manifold.

Proof: One uses the expression for y^{-1} for the c.t. Φ_1 given in (III.3.2), and then y^{-1} for c.t. Φ_2 given in (III.3.4). □

III.4.2 Proposition: The right action generators ${}^{R}X^{a}_{st}$ of SO(n) given in (III.4.1) transported in the body are

$$ {}^{b}X^{a}_{st} = \begin{cases} -\tilde{\mathcal{L}}_{st} & \text{for } s<t=1,2,3 \\ -\sum_{m=1}^{3}\sum_{\ell=m+1}^{3} {}^{L}\Theta^{b(\omega')}_{st,m\ell}(\beta)\tilde{\mathcal{L}}_{m\ell} + \tilde{\phi}_{st} & \text{for } s=1,2,3 \;\; t=4,5,\dots,n \\ 0 & \text{for } s<t=4,5,\dots,n \end{cases} $$

where ${}^{L}\Theta^{b(\omega')}_{st,m\ell}(\beta) := \sum_{k=1}^{3}\sum_{\tilde{m}=1}^{3}\sum_{\tilde{\ell}=\tilde{m}+1}^{3}\sigma'_{sk}(\omega')\,{}^{L}\Theta_{kt,\tilde{m}\tilde{\ell}}(\beta)^{t}\,d_{\tilde{m}\tilde{\ell},m\ell}(\omega')$

for $s=1,2,3$ $\quad t=4,5,\dots,n$ $\quad m<1=1,2,3$ and

$$ \tilde{\phi}_{st} := \sum_{\ell=1}^{3}\sigma'_{s\ell}(\omega')\,{}^{L}\Theta_{\ell t,\sigma}(\beta)\frac{\partial}{\partial\beta_\sigma} \quad \text{for } s=1,2,3 \;\; t=4,5,\dots,n. $$

Proof:

$$
{}^{b}X^{a}_{st} := \begin{cases} \sum_{v=1}^{3}\sum_{g=v+1}^{3}\sum_{\tilde{s}=1}^{n}\sum_{\tilde{t}=\tilde{s}+1}^{n} {}^{t}d^{-1}_{st,vg}(\omega)\,{}^{t}d^{-1}_{vg,\tilde{s}\tilde{t}}(\beta)\,{}^{R}X^{a}_{\tilde{s}\tilde{t}} & \text{for } s<t=1,2,3 \\ \sum_{v=1}^{3}\sum_{\tilde{s}=1}^{n}\sum_{\tilde{t}=\tilde{s}+1}^{n} \sigma'_{sv}(\omega)\,{}^{t}d^{-1}_{vt,\tilde{s}\tilde{t}}(\beta)\,{}^{R}X^{a}_{\tilde{s}\tilde{t}} & \text{for } s=1,2,3 \;\; t=4,5,\dots,n \\ 0 & \text{for } s<t=4,5,\dots,n \end{cases}
$$

and we use (I.1.11) and (I.2.3). □

III.4.3 Proposition: The left action generators of $SO(3)$ acting on the manifold $gl_+(3,\mathbb{R})\times\left(SO(n)\,|\,SO(3)\times SO(n-3)\right)$, expressed through the new coordinates after the coordinate transformation $\Phi = \Phi_2\circ\Phi_1$, are

$$
{}^{L}\chi^{a}_{lk} = -\sum_{g=1}^{3}\sum_{v=g+1}^{3} {}^{t}d^{-1}_{lk,gv}(\omega)\,L_{gv}\,, \qquad l<k=1,2,3
$$

where $L_{gv} := \sum_{\mu=1}^{3} {}^{R}\Theta_{gv,\mu}(\omega)\,\frac{\partial}{\partial\omega_\mu}\,, \quad g<v=1,2,3\,.$

Proof: After the c.t. Φ_1 the left action $SO(3)$ generators are defined on $gl_+(3,\mathbb{R})$ manifold parametrized by ξ'_{ij}, and after the c.t. Φ_2 one obtains the result in the same way as in (II.3.2) only for $n = 3$. □

III.5. The kinetic energy and the velocity field in $gl_+(3,\mathbb{R})\times SO(n)$ scheme

Equipped with the inverse Jacobi matrix for the coordinate transformation Φ_1 we calculate the complete kinetic energy for the relative motion of n+1 particles in new coordinates.

III.5.1 Proposition: The qunatum mechanical kinetic energy on $\mathbb{R}^{3n}$ is given after the c.t. Φ_1 (see (III.1.2)) in the new set of coordinates

$$T \sim \sum_{j=1}^{3}\sum_{l=1}^{3} \frac{\partial^2}{\partial \xi'^2_{jl}}$$

$$+ \sum_{m=1}^{3}\sum_{m'=1}^{3}\sum_{k=4}^{n}\sum_{i=1}^{3} \xi^{1-1}_{mi}\,\xi^{1-1}_{m'i}\;\widetilde{\phi}_{m'k}\;\widetilde{\phi}_{mk}$$

$$+ \sum_{l=1}^{3}\sum_{l'=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m=1}^{3}\sum_{m'=1}^{3}\sum_{\hat{m}=1}^{3}\sum_{i=1}^{3} \xi^{1-1}_{mi}\,\xi^{1-1}_{m'i}\,{}^{L}\Theta_{m'k,yl}(\beta)\,{}^{L}\Theta_{mk,\hat{m}l'}(\beta)\sum_{j=1}^{3}\xi'_{jy}\frac{\partial}{\partial \xi'_{jl}}\sum_{j'=1}^{3}\xi'_{j'\hat{m}}\frac{\partial}{\partial \xi'_{j'l'}}$$

$$-2\sum_{l=1}^{3}\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m'=1}^{3}\sum_{i=1}^{3} \xi^{1-1}_{mi}\,\xi^{1-1}_{m'i}\,{}^{L}\Theta_{m'k,yl}(\beta)\sum_{j=1}^{3}\xi'_{jy}\frac{\partial}{\partial \xi'_{jl}}\;\widetilde{\phi}_{mk}$$

$$+\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{l'=1}^{3}\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m'=1}^{3}\sum_{\hat{m}=1}^{3} {}^{L}\Theta_{m'k,yl}(\beta)\,{}^{L}\Theta_{mk,\hat{m}l'}(\beta)\,\xi^{1-1}_{mi}\sum_{j=1}^{3}\xi'_{jy}\left[\frac{\partial}{\partial \xi'_{jl}}\,\xi^{1-1}_{mi}\right]\sum_{j'=1}^{3}\xi'_{j'\hat{m}}\frac{\partial}{\partial \xi'_{j'l'}}$$

$$-\sum_{i=1}^{3}\sum_{l=1}^{3}\sum_{l'=1}^{3}\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{y=1}^{3}\sum_{m'=1}^{3} {}^{L}\Theta_{m'k,yl}(\beta)\,\xi^{1-1}_{m'i}\sum_{j=1}^{3}\xi'_{jy}\left[\frac{\partial}{\partial \xi'_{jl}}\,\xi^{1-1}_{mi}\right]\;\widetilde{\phi}_{mk}$$

$$-\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{l'=1}^{3}\sum_{m'=1}^{3}\sum_{\hat{m}=1}^{3}\sum_{i=1}^{3} \xi^{1-1}_{mi}\,\xi^{1-1}_{m'i}\left[\widetilde{\phi}_{m'k}\,{}^{L}\Theta_{mk,\hat{m}l'}(\beta)\right]\sum_{j=1}^{3}\xi'_{j\hat{m}}\frac{\partial}{\partial \xi'_{jl'}}$$

$$-\sum_{m'=1}^{3}\sum_{k=4}^{n}\sum_{k'=4}^{n}\sum_{l'=1}^{3}\sum_{m=1}^{3}\sum_{\hat{m}=1}^{3}\sum_{i=1}^{3} \xi^{1-1}_{mi}\,\xi^{1-1}_{m'i}\,{}^{L}\Theta_{m'k,k'k}(\beta)\,{}^{L}\Theta_{mk',\hat{m}l'}(\beta)\sum_{j=1}^{3}\xi'_{j\hat{m}}\frac{\partial}{\partial \xi'_{jl'}}$$

$$+\sum_{m'=1}^{3}\sum_{m=1}^{3}\sum_{k=4}^{n}\sum_{k'=4}^{n}\sum_{i=1}^{3} \xi^{1-1}_{mi}\,\xi^{1-1}_{m'i}\,{}^{L}\Theta_{m'k,k'k}(\beta)\;\widetilde{\phi}_{mk'}$$

where $\widetilde{\phi}_{mk} := -\sum_{\alpha=1}^{3n-9} {}^{L}\Theta_{mk,\alpha}(\beta)\frac{\partial}{\partial \beta_\alpha} \qquad m<k=1,2,\dots,n$

and ${}^{L}\Theta_{m'<k',m<k}(\beta) := \sum_{\mu=1}^{\frac{n(n-1)}{2}} {}^{L}\Theta_{m'<k',\mu}(\beta)\,P_{\mu,m<k}$, $P_{\mu,m<k} := \delta_{\mu,m<k}$

$m, k, m', k' = 1,2,\dots,n$. □

We are not going to discuss that rather complicated expression obtained in (III.5.1) because we use it only as an intermediate result. What we are really interested in is the question whether the kinetic energy in new coordinates after the c.t. Φ separates into $g\mathcal{L}_+(3,\mathbb{R})$ collective and intrinsic parts. For this goal it is only important to calculate all the quadratic terms in momenta in the expression for T given in (III.5.1) after the c.t. Φ_2. This can be easily done making use of (III.3.4).

III.5.2 Proposition: The quantum mechanical expression for the kinetic energy quadratic in momenta under the c.t. Φ defined in (III.1.3)

$$(\xi_{is} \mid i=1,2,3 \quad s=1,2,\ldots,n) \xrightarrow{\Phi} (\omega_\mu, \mathring{\xi}_i, \omega'_{\mu'}, \beta_\tau \mid \mu,\mu'=1,2,3 \; ; \; \tau=1,2,\ldots,3n-9)$$

takes the form

$$T_{(Q)} = \frac{1}{2m} \sum_{g=1}^{3} \sum_{v=g+1}^{3} \frac{\mathring{\xi}_g^2 + \mathring{\xi}_v^2}{(\mathring{\xi}_g^2 - \mathring{\xi}_v^2)^2} \hat{L}_{gv}^2$$

$$+ \frac{1}{2m} \sum_{g=1}^{3} \sum_{v=g+1}^{3} \frac{\mathring{\xi}_g^2 + \mathring{\xi}_v^2}{(\mathring{\xi}_g^2 - \mathring{\xi}_v^2)^2} \hat{\tilde{\mathcal{L}}}_{gv}^2$$

$$+ \frac{1}{2m} \sum_{g=1}^{3} \sum_{v=g+1}^{3} \frac{4\,\mathring{\xi}_g \mathring{\xi}_v}{(\mathring{\xi}_g^2 - \mathring{\xi}_v^2)^2} \hat{L}_{gv} \hat{\tilde{\mathcal{L}}}_{gv}$$

$$+ \frac{1}{2m} \sum_{v=1}^{3} \hat{t}_v^2$$

$$+ \frac{1}{2m} \sum_{k=4}^{n} \sum_{i=1}^{3} \mathring{\xi}_i^{-2} \hat{\tilde{\phi}}_{ik}^2$$

$$+ \frac{1}{2m} \sum_{k=4}^{n} \sum_{i=1}^{3} \sum_{g=1}^{3} \sum_{v=g+1}^{3} 2\,\mathring{\xi}_i^{-2} L\Theta^{b(\omega')}_{ik,gv}(\beta) \hat{\tilde{\mathcal{L}}}_{gv} \hat{\tilde{\phi}}_{ik}$$

$$+\frac{1}{2m}\sum_{k=4}^{n}\sum_{i=1}^{3}\sum_{g=1}^{3}\sum_{\nu=g+1}^{3}\sum_{\tilde{g}=1}^{3}\sum_{\tilde{\nu}=\tilde{g}+1}^{3}\overset{\circ}{\xi}{}_i^{-2}\,{}^{L}\Theta^{b(\omega')}_{ik,g\nu}(\beta)\,{}^{L}\Theta^{b(\omega')}_{ik,\tilde{g}\tilde{\nu}}(\beta)\,\hat{\mathcal{L}}_{g\nu}\hat{\mathcal{L}}_{\tilde{g}\tilde{\nu}}$$

where

$$\hat{L}_{g\nu} := -i\hbar\sum_{\mu=1}^{3}{}^{R}\Theta_{g\nu,\mu}(\omega)\frac{\partial}{\partial\omega_\mu} \qquad g<\nu=1,2,3$$

$$\hat{t}_\nu := -i\hbar\frac{\partial}{\partial\overset{\circ}{\xi}_\nu} \qquad \nu=1,2,3$$

$$\hat{\tilde{\mathcal{L}}}_{g\nu} := i\hbar\sum_{\mu'=1}^{3}{}^{L}\Theta_{g\nu,\mu'}(\omega')\frac{\partial}{\partial\omega'_{\mu'}} \qquad g<\nu=1,2,3$$

$$\hat{\tilde{\phi}}_{ik} := i\hbar\sum_{m=1}^{3}\sigma'_{im}(\omega')\sum_{\alpha=1}^{3n-9}{}^{L}\Theta_{mk,\alpha}(\beta)\frac{\partial}{\partial\beta_\alpha} \qquad i=1,2,3 \quad k=4,5,\dots,n$$

$${}^{L}\Theta^{b(\omega')}_{ik,g\nu}(\beta) := \sum_{m=1}^{3}\sum_{\ell=1}^{3}\sum_{a=1}^{3}\sigma'_{im}(\omega')\sigma'_{g\ell}(\omega')\sigma'_{\nu a}(\omega')\,{}^{L}\tilde{\Theta}_{mk,\ell a}(\beta) \qquad i=1,2,3;$$

$$g<\nu=1,2,3;\ k=4,5,\dots,n. \qquad \square$$

Let us discuss the expression obtained in (III.5.2). Through the c.t. ϕ we defined vortex spin $\hat{\mathcal{L}}$ on an $SO(3)$ manifold and the intrinsic momenta $\hat{\phi}$ so that they contain no partial derivatives w.r. t. the $gl_+(3,\mathbb{R})$ parameters, or more exactly $SO(3)$ parameters which define the vortex spin $\hat{\mathcal{L}}$. All that can be seen in (III.4.1), and that is the fact which we exactly missed in the $SO(3)\times SO(n)$ scheme, where both vortex spin $\hat{\mathcal{L}}$ and the intrinsic operators $\hat{\phi}$ were defined on the right coset SO(n)/SO(n-3) (see (II.3.5) and compare with (III.4.1) and (III.4.2)). From (III.5.2) we see that the complete separation has not been achieved: in the sixth and seventh line in $T_{(Q)}$ the coupling terms appear. The matrix block different from zero which causes those terms to appear is ${}^{L}\Theta^{b(\omega')}_{mk,\ell i}(\beta)$ or equivalently ${}^{L}\Theta_{mk,\ell i}(\beta)$ for m = 1,2,3 k = 4,5,..,n $1<i=1,2,3$ (see (III.4.2)). If we would postulate that ${}^{L}\Theta_{mk,\ell i}(\beta)=0$, what is equivalent to the constraints ${}^{L}\mathcal{T}_{mk,\ell i}(\beta)=0$ for the same range of indices (see (I.1.4) and (I.2.3)) we would obtain the complete separation. This is not surprising if we remember the content of the section III.2 where we obtained the total separation due to the same constraints postulated from the very beginning. In this section we calculated exactly the kinetic energy due to the exact expression for M^{-1}

for c.t. Φ , and we conclude that the constraints are necessary and sufficient if one wants to get the total separation of kinetic energy into intrinsic and collective parts w.r. t. the $g\mathcal{L}_+(3,\mathbb{R})$ collectivity.
It should be stressed that the constraints ${}^{L}\Theta_{mk,ei}(\beta)=0$ for $m = 1,2,3$ $k = 4,5\ldots,n$ $1 < i = 1,2,3$ do not follow from the Lie-group theory.

Now we can try to define the linear velocity field or more exactly its collective part as in section III.2. Making use of γ_g^{-1} for both c.t. Φ_1 and Φ_2 we obtain the candidate for π_{is}^{coll}

$$
\begin{aligned}
"\pi_{is}^{coll}" &= \sum_{l=1}^{3} \chi^{S}_{il}\,\xi_{ls} \\
&+ \sum_{l=1}^{3} \chi^{A}_{il}\,\xi_{ls} \\
&- \sum_{l=1}^{3}\sum_{k=4}^{n}\sum_{g=1}^{3}\sum_{v=g+1}^{3} \overset{\circ}{\xi}{}^{-1}_{l}\,\sigma_{il}(\omega)\,{}^{L}\Theta^{b(\omega')}_{lk,gv}(\beta)\,\sigma'_{ks}(\beta)\,\tilde{\mathcal{L}}_{gv}
\end{aligned}
$$

where

$$
\begin{aligned}
\chi^{S}_{il} :&= \sum_{g=1}^{3}\sum_{v=g+1}^{n} \left(\sigma_{ig}(\omega)\,\sigma_{lv}(\omega) + \sigma_{lg}(\omega)\,\sigma_{iv}(\omega)\right) \frac{1}{\xi_v^2 - \xi_g^2} \\
&\times\left[L_{gv} + \frac{\overset{\circ}{\xi}{}^2_g + \overset{\circ}{\xi}{}^2_v}{2\,\overset{\circ}{\xi}_v\,\overset{\circ}{\xi}_g}\,\tilde{\mathcal{L}}_{gv}\right] \\
&+ \sum_{w=1}^{3} \sigma_{iw}(\omega)\,\overset{\circ}{\xi}{}^{-1}_{w}\,\sigma_{lw}(\omega)\,\pi_w\,, \qquad \chi^{S}_{il} = \chi^{S}_{li}
\end{aligned}
$$

and

$$
\chi^{A}_{il} := \sum_{g=1}^{3}\sum_{v=g+1}^{3} \frac{1}{2}\left(\sigma_{iv}(\omega)\,\sigma_{lg}(\omega) - \sigma_{lv}(\omega)\,\sigma_{ig}(\omega)\right) \frac{1}{\xi_v\,\xi_g}\,\tilde{\mathcal{L}}_{gv}\,, \qquad \chi^{A}_{il} = -\chi^{A}_{li}
$$

and where L_{gv}, $\tilde{\mathcal{L}}_{gv}$ and π_w are the classical momenta which take the same form as the corresponding differential operators with partial derivatives replaced by the canonical momenta. The last line in $"\pi_{is}^{coll}"$ proportional to ${}^{L}\Theta_{mk,ei}(\beta)$ $m = 1,2,3$ $k = 4,5\ldots,n$ $1 < i = 1,2,3$, spoils the separation. That is the result which we clearly expected.

IV. The $U(3) \times U(n)$ scheme

IV.0 Introduction

The content of this and the following chapter should be seen in the broader physical frame of the theory of nuclear collective motion through representations of the symplectic group $Sp(6n, \mathbb{R})$ of linear canonical transformations of the n+1-nucleon system[29]. In this theory the state description is chosen by square-integrable analytic functions $F(x)$ of $3n$ complex variables* $(x_{is} | i = 1,2,3 \quad s = 1,2,\ldots,n)$ in Bargmann[49] space. The oscillator creation and annihilation operators X_{is} and D_{is} act on these functions as

$$(X_{is}F)(x) = x_{is}F(x)$$

$$(D_{is}F)(x) = \frac{\partial}{\partial x_{is}} F(x)$$

The generators of $Sp(6n, \mathbb{R})$ become

$$A^+_{is,jt} = \tfrac{1}{2} X_{is} X_{jt}, \quad A_{is,jt} = \tfrac{1}{2} D_{is} D_{jt}, \quad C_{is,jt} = \tfrac{1}{4}\left(X_{is} D_{jt} + D_{jt} X_{is}\right).$$

The collective and the intrinsic symplectic group $Sp(6,\mathbb{R})$ and $Sp(2n, \mathbb{R})$ are the groups generated by

$$Sp(6,\mathbb{R}): \quad A^+_{ij} = \sum_{s=1}^{n} A^+_{is,js}, \quad A_{ij} = \sum_{s=1}^{n} A_{is,js}, \quad C_{ij} = \sum_{s=1}^{n} C_{is,js}$$

$$Sp(2n,\mathbb{R}): \quad A^+_{st} = \sum_{j=1}^{3} A^+_{js,jt}, \quad A_{st} = \sum_{j=1}^{3} A_{js,jt}, \quad C_{st} = \sum_{j=1}^{3} C_{js,jt}.$$

The collective subgroup $U(3)$ is generated by C_{ij} and $O(3, \mathbb{R})$ by $C_{ij} - C_{ji}$, and the intrinsic subgroup $U(n)$ is generated by C_{st} and $O(n, \mathbb{R})$ by $C_{st} - C_{ts}$.

The collective and intrinsic subgroups commute with one another according to Fig. 1

*The center of mass should be considered as being separated.

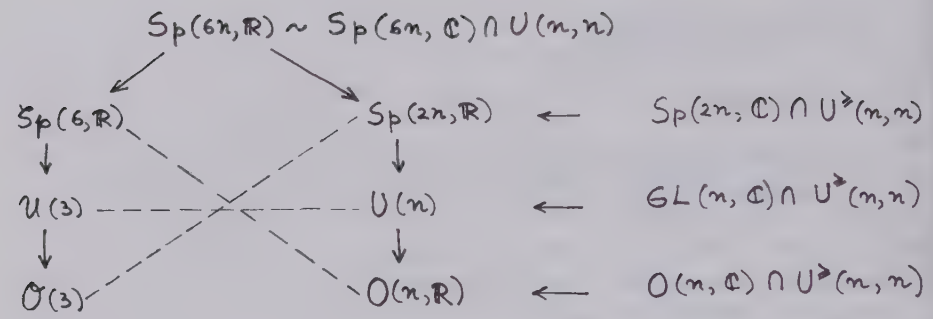


Fig. 1: Subgroup and subsemigroup relations (full lines with arrows) and commutativity (dashed lines) for collective and intrinsic (semi-) groups.

For the fixed intrinsic group the maximal subgroup of $Sp(6,\mathbb{R})$ which commutes with it is connected to it by the dashed line in Fig. 1. This statement can be put also the other way around. Let us call such a couple "maximal commuting pair". For a microscopic theory of collective motion Vanagas proposed intrinsic groups as relevant: he defined H_{coll} as the part of the full hamiltonian which is invariant under the intrinsic group $O(n,\mathbb{R})$ [28]. In various papers it was shown that the hamiltonian with two-body interactions of Gaussian type is contained in the operator algebra of the symplectic group, provided the corresponding representation is extended to the semigroup $S_p(2n,\mathbb{C}) \cap U^{\geq}(n,n)$ [50,51]. The semigroup $U^{\geq}(n,n)$ consists of all length-increasing transformations with respect to the hermitian form underlying U(n,n), it contains the group $S_p(2n,\mathbb{R}) \sim S_p(2n,\mathbb{C}) \cap U(n,n)$. Subsemigroups are indicated in Fig. 1.

We propose here U(n)-collectivity by demanding that H_{coll} be invariant under the intrinsic group U(n). Then, since U(n) contains the generator $H_0 := \sum_{i=1}^{3} \sum_{s=1}^{n} C_{is,is}$ which counts the total number of quanta, the states belong to an irreducible representation $[N]$ of the group U(1) generated by H_0. This fixes the

irreducible representation of the group U(3n) generated by all the $C_{is,jt}$ to be $[N\,0^{3n-1}]$, and complementarity[31] applied to the intrinsic and collective groups U(n) and $\mathcal{U}(3)$ tells us that the irreducible representations must be of the form $[h_1 h_2 h_3\, 0^{n-3}]$ and $[h_1\, h_2\, h_3]$ respectively[27]. We can put all this in a simple statement: for U(n)-collectivity, H_{coll} is studied in the enveloping algebra[52] of the corresponding collective dynamical group $\mathcal{U}(3)$. This statement is also good for phenomenological models based on U(n) collectivity.

Until now we discussed the quantum-mechanical aspect of the problem. We should mention also the classical aspect. Symplectic groups are groups of linear canonical transformations in phase space. If we keep to the complex realization of the problem, then the point in the phase space is represented by $\begin{bmatrix} x_{is} \\ \overline{x}_{is} \end{bmatrix}$ $i = 1,2,3$, $s = 1,2,\ldots,n$ where $x_{is} := (\xi_{is} - i\pi_{is})$ and ξ_{is} and π_{is} are as in (II.2). Then the maximal commuting pairs in Fig. 1 act in the 6n dimensional phase space so that the collective group act on phase space index "i" which runs twice from 1 to 3, and intrinsic group on the relative vector index "s". In all three cases from Fig. 1 the action can be written in such a way that the collective group acts via left group action and intrinsic group via right action on a certain matrix which gives the configuration of n relativ vectors of n+1 particles in phase space. Note that for this statement it is a necessary condition for the collective and intrinsic groups to commute. Through the action of $\mathcal{U}(3) \times U(n)$ on representative points we span the phase space for our problem and we get rid of redundant coordinates doing the orbit analysis. In the frame of the orbit analysis, using the fact that H_{coll} is U(n) invariant, we define the collective phase space which is represented by dissociated collective orbits. The kinematics and the dynamics on the single collective orbit are given by collective classical observables defined as the functions of the collective orbit coordinates.

By the orbit analysis in phase space we use again as the most important input the complementarity which here means that the collective and the intrinsic tensor in the $U(3) \times U(n)$ scheme have the same diagonal form (see (IV.1.5), (IV.1.9) and (IV.1.13)). This is an important link between the particle lable space and the 6-dimensional phase space.

The collective groups related to the intrinsic groups was first studied in ref. 29,35 and especially the maximal commuting pair $Sp(6,R)$ and O(n) in ref. 30, 53, 41 and 47.

IV.1. Orbits in phase space under the direct product of the collective and intrinsic groups

In the reference 35 the orbit analysis in phase space under the action of $\mathcal{U}(s)\times U(n)$ group has been studied. We bring it here reformulated.
Consider a system of A = n + 1 particles moving in 3-dimensional space and denote by ξ_{is} , π_{is} , i = 1,2,3, s = 1,2,...,n the n relative Jacobi position and momentum coordinates.

IV.1.1 Definition: A point in 6n-dimensional phase space $\mathcal{P}$ is characterized by a 6 x 2n matrix $\left(\begin{array}{c|c} x_{is} & 0 \\ \hline 0 & \overline{x_{is}} \end{array}\right)$

where $x_{is} := \frac{1}{\sqrt{2}}\left(\xi_{is} - i\pi_{is}\right)$.

IV.1.2 Definition: The action of the group $\mathcal{U}(3)\times U(n)$ with elements $(\tilde{u},\tilde{g})$ where $\tilde{u}\in\mathcal{U}(3)$, $\tilde{g}\in U(n)$ on phase space is given by the linear canonical transformation $\tilde{G}\times\mathcal{P}\to\mathcal{P}$

$$\begin{pmatrix} \tilde{u}^+ & 0 \\ 0 & \overline{\tilde{u}^+} \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & \overline{x} \end{pmatrix}\begin{pmatrix} \tilde{g} & 0 \\ 0 & \overline{\tilde{g}} \end{pmatrix} = \begin{pmatrix} \tilde{u}^+ x\tilde{g} & 0 \\ 0 & \overline{\tilde{u}^+ x\tilde{g}} \end{pmatrix}.$$

Once the orbit analysis is implemented according to $x \to \tilde{u}^+ x \tilde{g}$ on $\mathcal{M}\simeq\mathbb{C}^{3n}$, it can immediately be extended to the full phase space $\mathcal{P}$.

IV.1.3 Definition: The collective tensor $\mathcal{T}(x)$ for $x\in\mathbb{C}^{3n}$ is given by

$$\mathcal{T}(x) := xx^+, \text{ or}$$

$$\mathcal{T}_{ij}(x) = -\overline{c_{ij}}$$

where

$$c_{ij} := -\sum_{s=1}^{n} \overline{x_{is}}\, x_{js}.$$

The collective tensor $\mathcal{T}(x)$ can be also written in the form

$$\mathcal{T}_{ij}(x) = \frac{1}{2}\left[\sum_{s=1}^{n}\xi_{is}\xi_{js} + \sum_{s=1}^{n}\pi_{is}\pi_{js} + i\sum_{s=1}^{n}\left(\xi_{is}\pi_{js} - \xi_{js}\pi_{is}\right)\right]$$

IV.1.4 Proposition: Define on classical phase space $\mathcal{P}$ the Poisson bracket

$\{\ ,\ \}_{\mathcal{P}} \equiv \{\ ,\ \}(x,\bar{x})$ by

$$\{F,G\}_{\mathcal{P}} = \sum_{i=1}^{3}\sum_{s=1}^{n}\left(\frac{\partial F}{\partial \bar{x}_{is}}\frac{\partial G}{\partial x_{is}} - \frac{\partial F}{\partial x_{is}}\frac{\partial G}{\partial \bar{x}_{is}}\right)$$

Then the fundamental Poisson brackets are

$$\{x_{is}, x_{jt}\}_{\mathcal{P}} = 0\ ,\quad \{\bar{x}_{is}, \bar{x}_{jt}\}_{\mathcal{P}} = 0,$$

$$\{\bar{x}_{is}, x_{jt}\}_{\mathcal{P}} = \delta_{ij}\delta_{st}\ ,$$

and the functions c_{ij} are the symplectic or Poisson generators of the group $\mathcal{U}(3)$,

$$\{c_{ij}, c_{kl}\}_{\mathcal{P}} = \delta_{jk}c_{il} - \delta_{il}c_{kj}.$$

Note that the Poisson bracket has been modified in a way which makes it equivalent to the commutation relations in quantum mechanics.

IV.1.5 Proposition: The collective tensor $\mathcal{T}(x)$ has the properties (1) under $\tilde{\mathcal{E}} = \mathcal{U}(3)\times U(n)$ with elements $(\tilde{u},\tilde{q})$, $\mathcal{T}(x)$ transforms $\mathcal{T}(\tilde{u}\times\tilde{q}) = \tilde{u}+\mathcal{T}(x)\tilde{u}$. (2) $\mathcal{T}^{+}(x) = \mathcal{T}(x)$ so that exists $u\in\mathcal{U}(3)$ which diagonalize $\mathcal{T}(x)$

$\mathcal{T}(x) = u\mu u^{+}$, where $\mu_{ij} = \mu_i\delta_{ij} =: \mathring{\mathcal{T}}_{ij}(x)$, $i,j = 1,2,3$.

$\mathcal{T}(x)$ is also positive semidefinite, so that $\mu_i \geq 0$. We are going to consider only the case when $\mu_i > 0$. (μ_i) can be chosen to be an ordered set in the same fashion as in $SO(3)\times SO(n)$ scheme (see (II.1)): $\mu_1 > \mu_2 > \mu_3 > 0$.

IV.1.6 Proposition: The stability group $\mathcal{H}_3' < \mathcal{U}(3)$ of $\mathring{\mathcal{T}}(x)$ is the diagonal subgroup $\mathcal{A}_3$ of $\mathcal{U}(3)$.

IV.1.7 Definition: The intrinsic tensor $T(x)$ is given by

$T(x) := x^{+}x$

with elements

$$T_{st}(x) =: c_{st}$$

$$= \sum_{i=1}^{3} x_{is}\,\overline{x_{it}}$$

which are n^2 symplectic generators of the group U(n) in phase space.

IV.1.8 Proposition: The intrinsic tensor T(x) has the properties
(1) under $\tilde{G} = U(3) \times U(n)$ with elements $(\tilde{u}, \tilde{g})$ T(x) transforms as

$$T(\tilde{u}^+ x \tilde{g}) = \tilde{g}^+ T(x) \tilde{g} \;;$$

(2) $T^+(x) = T(x)$.

IV.1.9 Proposition: For points $x \in \mathbb{C}^{3n}$ with strictly positive $\mathcal{T}(x)$ there exists an element $g \in U(n)$ such that

$$T(x) = g^+ \mathring{T}(x)\, g \,,$$

$$\mathring{T}(x) := \left(\begin{array}{c|c} \mu & 0 \\ \hline 0 & 0 \end{array}\right).$$

Proof: Define the special point $\mathring{x} \in \mathbb{C}^{3n}$ by

$$\mathring{x} := (\mathring{t} \mid 0) \,, \quad \mathring{t}_{ij} = \mu_i^{\frac{1}{2}} \delta_{ij} \quad i,j = 1,2,3.$$

Then $\mathcal{T}(\mathring{x}) = \mathring{\mathcal{T}}(x)$. Define $3 \times n$ matrix g' by

$$g' = \mathring{t}^{-1} u^+ x \qquad \text{to find}$$

$$g'(g')^+ = \mathring{t}^{-1} u^+ x x^+ u \mathring{t} = \mathring{t}^{-1} \mathring{\mathcal{T}}(x) \mathring{t} = I_3 .$$

Hence the matrix g' can be supplemented by n - 3 rows g'' to yield an $n \times n$ matrix $g \in U(n)$ such that

$$x = u \mathring{x} g .$$

For the tensor T(x) one now finds $T(x) = g^+ T(\mathring{x})\, g$ with the form of $\mathring{T}(x) = T(\mathring{x})$ given above. □

IV.1.10 Proposition: The stability group $H' < U(n)$ of $\mathring{T}(x)$ is the subgroup $A_3 \times U(n-3)$ of $U(n)$.

IV.1.11 Proposition: Under $\tilde{G} = \mathcal{U}(3) \times U(n)$ the phase space decomposes for those points which fulfil $\mathcal{T}(x) > 0$, into transversals characterized by the representative point $\mathring{x}$, and coset spaces of the form

$$\mathcal{U}(3) \times U(n) \,/\, diag\,(\mathcal{H}_3', H_3') \times U(n-3) \quad \text{where} \quad diag\,(\mathcal{H}_3', H_3')$$

is the diagonal subgroup with elements (h,h) of $\mathcal{H}_3' \times H_3'$ and where $H_3' = A_3$.

Proof: We have to determine the stability group H of the representative point $\mathring{x}$. Clearly H must be a subgroup of the two groups $\mathcal{H}_3' \cap U(n)$ and $\mathcal{U}(3) \cap H_n'$ which are the full stability groups of $\mathcal{T}(\mathring{x})$ and $T(\mathring{x})$, respectively. Then clearly $H < \mathcal{H}_3' \times H_n'$ where $H_n' = H_3' \times U(n-3)$ and we know that $\mathcal{H}_3'$ and H_3' are ismorphic and are in this case $\mathcal{A}_3$ and A_3, respectively. Inspection of the action of these groups on $\mathring{x}$ shows that $H = diag\,(\mathcal{H}_3', H_3') \times U(n-3)$ where $diag\,(\mathcal{H}_3', H_3') \equiv$ $\equiv diag(\mathcal{A}_3, A_3)$ is isomorphic to $\mathcal{A}_3$ or A_3. □
Note that $diag\,(\mathcal{A}_3, A_3)$ can be taken to be either the subgroup of $\mathcal{U}(3)$ or of $U(n)$ via $\mathcal{A}_3$ or A_3.
Now we look at what happens with respect to the U(n) action on the phase space for $\mathcal{T}(x) > 0$.

IV.1.12 Proposition: Under U(n) action the phase space decomposes into the transversal characterized by both $\mathring{x}$ and a left coset space $\mathcal{U}(3)/\mathcal{A}_3$, and into the orbit which is the right coset space U(n)/U(n-3

Proof: The transversal of the action of U(n) by definition contains a representative point from the orbit under the $\mathcal{U}(3) \times U(n)$ action. We get the orbit of U(n) by decomposing the orbit of $\mathcal{U}(3) \times U(n)$. This orbit is characterized by the pair of coset spaces $\mathcal{U}(3)/\mathcal{A}_3$ and U(n)/U(n-3). It follows that the transversal of the U(n) action is characterized by $\mathring{x}$ and $\mathcal{U}(3)/\mathcal{A}_3$. □

IV.1.13 Proposition: The Casimir elements of $\mathcal{U}(3)$ and U(n) defined for $\mathcal{U}(3)$

$$w_m = tr\left((\overline{T}(x))^m\right) \quad m = 1,2,\dots$$ and for $U(n)$ by

$$w_\ell = tr\left((T(x))^\ell\right) \quad \ell = 1,2,\dots$$ (see ref.52 p. 255)

are both functions of the matrix $\mathring{t}$.

Proof: Recall (IV.1.5) and (IV.1.9). □

IV.1.14 Definition: For those points which fulfil $\mathcal{T}(x) > 0$ one can introduce the new set of coordinates via the relation

(1) $$x_{is} = \sum_{q=1}^{3} \mu_{iq}(\omega)\, \mathring{x}_q\, w_{qs}(\beta), \quad i = 1,2,3 \quad s = 1,2,\dots,n$$

where $\mu(\omega) \in \mathcal{U}(3)/A_3$ left coset space and $w(\beta) \in U(n) \mid U(n-3)$ right coset space: $\left(\omega_\tau, \mathring{x}_q, \beta_\sigma \mid \tau = 1,2,\dots,6 \quad q = 1,2,3 \quad \sigma = 1,2,\dots,6n-9\right)$ or

(2) $$x_{is} = \sum_{q=1}^{3} \mu_{iq}(\delta)\, \mathring{x}_q\, w_{qs}(\beta), \quad i = 1,2,3 \quad s = 1,2,\dots,n$$

where $\mu(\delta) \in \mathcal{U}(3)$ and $w(\beta) \in U(n) / A_3 \times U(n-3)$ right coset space:

$\left(\delta_\mu, \mathring{x}_q, \beta_\sigma \cdot \mu = 1,2,\dots,9 \quad q = 1,2,3 \quad \sigma = 1,2,\dots,6n-12\right)$.

Note that other orbits for $\mathcal{T} \geq 0$ which are listed in ref. 35 are less then 6n-dimensional.

The case (1) from (IV.1.14) is related to the following statement. Since the collective Hamiltonian H_{coll} is assumed to be invariant under U(n), its trajectories on phase space should be trivial on the orbits of this group. It follows that the collective dynamics and corresponding trajectories lie on the transversal for the action of U(n). This transversal is determined in (IV.1.12). The part $\mathring{x}$ of this transversal from the proposition (IV.1.13) is determined by the Casimir elements of U(n) and hence is a constant of motion. From all that yields that the collective dynamics for an U(n) invariant Hamiltonian can be described on the coset space $\mathcal{U}(3)/A_3$. The case (2) is related to the following statement. For all those models which consider the $\mathcal{U}(3)$ as its

dynamical group[54,5,27], what is always compatible with the U(n) invariance and $\mathring{x}_q$ fixed, one can define the collective coordinates as those which get changed under the $\mathcal{U}(3)$ left action. In the considerations which follow we still leave $\mathring{x}_q$ variable. They can be fixed when needed.

In order to give some physical interpretation from the collective tensor $\mathcal{T}(x)$ let us write it as a function of real coordinates and momenta

$$\mathcal{T}_{ij}(x) = \frac{1}{2}\left\{ \sum_{s=1}^{n} \xi_{is}\,\xi_{js} + \sum_{s=1}^{n} \pi_{is}\,\pi_{js} + i \sum_{s=1}^{n} (\xi_{is}\,\pi_{js} - \xi_{js}\,\pi_{is}) \right\}$$

$$= \frac{1}{2}\left\{ \mathcal{T}_{ij}(\xi) + \sum_{s=1}^{n} \pi_{is}\,\pi_{js} + i\,{}^{L}\chi^{a}_{ij} \right\} \qquad \text{(II.3.1)}$$

Now let us suppose that $\mathcal{T}(x)$ is quantized. The imaginary part of $\mathcal{T}(x)$ is clearly the angular momentum in the laboratory frame. In the real part of $\mathcal{T}(x)$ the collective tensor $\mathcal{T}(\xi)$ (see p. 38) is straight forward correlated with its traceless form which is the quadrupole mass moment operator

$$\left(Q^{(2)}\right)_{ij} := \mathcal{T}_{ij}(\xi) - \frac{1}{3}\,\mathrm{tr}\,\mathcal{T}(\xi)\,\delta_{ij}.$$

But in the real part of $\mathcal{T}(x)$ there appears also $\sum_{s=1}^{n} \pi_{is}\,\pi_{js}$. The matrix elements of the traceless part of $\sum_{s=1}^{n} \pi_{is}\,\pi_{js}$ within the oscillator shell are the same as those of $Q^{(2)}$ (see ref. 54 p. 133, ref. 5 p. 576 and also ref. 55 p. 138).

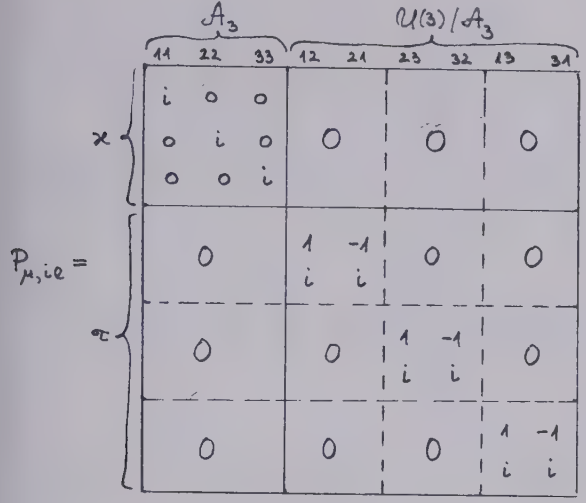
From now on we are going to study the coordinate transformation

$$\left(x_{is},\ \overline{x_{is}}\ \middle|\ i=1,2,3\ \ s=1,2,\ldots,n\right) \to \left(\omega_\tau,\ \mathring{x}_i,\ \beta_\sigma\ \middle|\ \tau=1,2,\ldots,6\ \ \sigma=1,2,\ldots,6n-9\right)$$

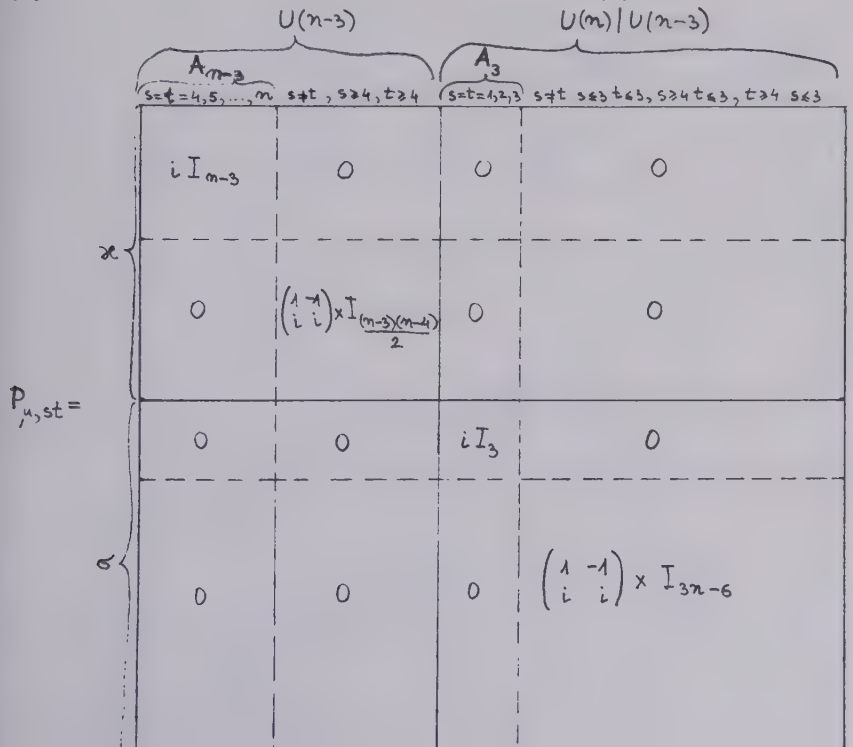
given in definition (IV.1.14)(1), and we give only the corresponding results for the case (2).

Let us write the P matrices defined in (I.3.2) which determine the relation between real and corresponding complex group parameters for the $\mathcal{U}(3)$ and U(n) group in the unitary scheme.

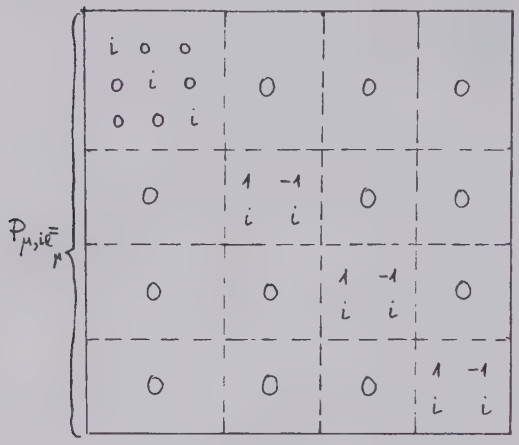
(1) IV.1.15 Definition: The matrix P for $\mathcal{U}(3)$ is



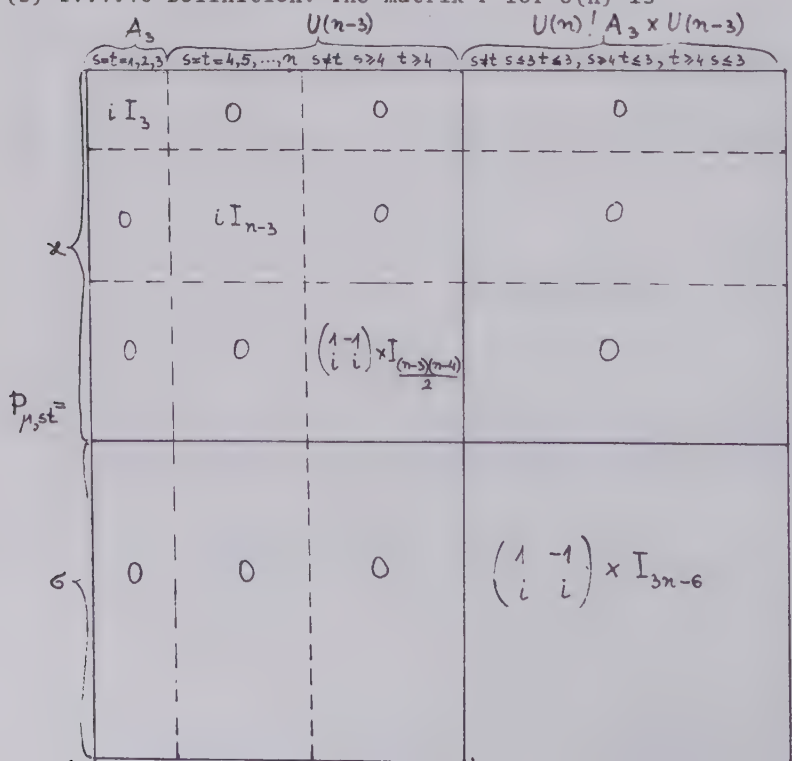
(1) IV.1.16 Definition: The matrix P for U(n) is



(2) IV.1.15 Definition: The matrix P for $\mathcal{U}(3)$ is



(2) IV.1.16 Definition: The matrix P for U(n) is



IV.2. The Jacobi matrix and its inverse

In (IV.3) we shall express the generators of the group $\mathcal{U}(3)\times U(n)$ on the orbit defined by $\mathring{x}_1 > \mathring{x}_2 > \mathring{x}_3 > 0$ through the coordinates ω and β . We also want (in (IV.4)) to determine how the Bargmann measure looks like when expressed through the coordinates $\omega(\xi), \mathring{x}$ and β . One way to achieve that is through the Jacobi matrix and its inverse. So, let us determine them.

(1) IV.2.1 Proposition: The Jacobi matrix for the transition

$$(a_{(is)}, b_{(is)} \mid a := \mathrm{Re}\, x\,,\ b := \mathrm{Im}\, x) \to (\omega_\alpha, \mathring{x}_\varrho, \beta_\sigma \mid \alpha = 1,\dots,6\ \ \varrho = 1,2,3\ \ \sigma = 1,\dots,6n-9) \equiv (\xi_\nu \mid \nu = 1,2,\dots,6n)$$

is given through its matrix elements

$$\mathcal{J}_{\nu,(is)} : \to \boxed{\left(\frac{\partial a}{\partial \xi}\right)_{\nu,(is)} \quad \left(\frac{\partial b}{\partial \xi}\right)_{\nu,(is)}} = \boxed{\frac{\partial a_{(is)}}{\partial \xi_\nu} \quad \frac{\partial b_{(is)}}{\partial \xi_\nu}}$$

and can be written as the product

$$\mathcal{J}_{\nu,(is)} = \sum_{k=1}^{3} \sum_{t=1}^{n} K_{\nu,kt}\, C_{kt,(is)} \quad , \text{ where}$$

$$C := \frac{1}{2}\begin{pmatrix} I_{3n} & -i I_{3n} \\ I_{3n} & i I_{3n} \end{pmatrix} \qquad \text{and}$$

$$K_{\nu,is} : \to \boxed{\left(\frac{\partial x}{\partial \xi}\right)_{\nu,is} \quad \left(\frac{\partial \bar{x}}{\partial \xi}\right)_{\nu,is}} = \boxed{\frac{\partial x_{is}}{\partial \xi_\nu} \quad \frac{\partial \bar{x}_{is}}{\partial \xi_\nu}}$$

(I) $$\left(\frac{\partial x}{\partial \omega}\right)_{\alpha,is} = \sum_{m=1}^{3} \sum_{\ell=1}^{3} u_{im}(\omega)\, {}^{R}\mathcal{I}_{\alpha,m\ell}(\omega)\, \mathring{x}_\ell\, w_{\ell s}(\beta),$$

$$\left(\frac{\partial \bar{x}}{\partial \omega}\right)_{\alpha,is} = \overline{\left(\frac{\partial x}{\partial \omega}\right)_{\alpha,is}}\ ;$$

(II) $$\left(\frac{\partial x}{\partial \beta}\right)_{\sigma,is} = \sum_{\ell=1}^{3} \sum_{\varrho=1}^{n} u_{i\ell}(\omega)\, \mathring{x}_\ell\, {}^{L}\mathcal{I}_{\sigma,\ell\varrho}(\beta)\, w_{\varrho s}(\beta),$$

$$\left(\frac{\partial \bar{x}}{\partial \beta}\right)_{\sigma,is} = \overline{\left(\frac{\partial x}{\partial \beta}\right)_{\sigma,is}}\ ;$$

(III) $$\left(\frac{\partial x}{\partial \mathring{x}}\right)_{q,is} = u_{iq}(\omega)\, w_{qs}(\beta),$$

$$\left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{q,is} = \overline{\left(\frac{\partial x}{\partial \mathring{x}}\right)_{q,is}}\ .$$

Proof: For (I), (II) and (III) we differentiate the relation

$$x_{is} = \sum_{\ell=1}^{3} u_{i\ell}(\omega)\, \mathring{x}_\ell\, w_{\ell s}(\beta), \quad u(\omega) \in \mathcal{U}(3) \,|\, \mathcal{A}_3, \quad w(\beta) \in U(n) \,|\, U(n-3)$$

with respect to the new coordinates ω, β and $\mathring{x}$, respectively. Determining (I) we use (I.3.15), and determining (II), (I.3.12). □

Now we invert Jacobi matrix, inverting its nontrivial part K.

(1) IV.2.2 Proposition: The inverse Jacobi matrix J^{-1} is given via K^{-1}

$$K^{-1}_{is,\nu} \to \begin{vmatrix} \left(\frac{\partial \xi}{\partial x}\right)_{is,\nu} \\ \left(\frac{\partial \xi}{\partial \bar{x}}\right)_{is,\nu} \end{vmatrix} = \begin{vmatrix} \frac{\partial \xi_\nu}{\partial x_{is}} \\ \frac{\partial \xi_\nu}{\partial \bar{x}_{is}} \end{vmatrix}$$

$$= \begin{vmatrix} \left(\frac{\partial \xi}{\partial x}\right)_{is,\nu} \\ \overline{\left(\frac{\partial \xi}{\partial x}\right)_{is,\nu}} \end{vmatrix}$$

$$\text{(I)} \quad \left(\frac{\partial \omega}{\partial x}\right)_{is,\pi} = \sum_{k=1}^{3} \sum_{\substack{j=1 \\ (k \neq j)}}^{3} \overline{u_{ik}(\omega)}\, \overline{w_{js}(\beta)}\, \frac{\mathring{x}_j}{\mathring{x}_j^2 - \mathring{x}_k^2}\, {}^{R}\Theta_{kj,\pi}(\omega)$$

$$\text{(II)} \quad \left(\frac{\partial \beta}{\partial x}\right)_{is,\sigma} = \sum_{k=1}^{3} \sum_{\substack{j=1 \\ (k \neq j)}}^{3} \overline{u_{ij}(\omega)}\, \overline{w_{ks}(\beta)}\, \frac{\mathring{x}_j}{\mathring{x}_j^2 - \mathring{x}_k^2}\, {}^{L}\Theta_{jk,\sigma}(\beta)$$

$$+ \sum_{j=1}^{3} u_{ij}(\omega)\, w_{js}(\beta)\, \tfrac{1}{2}\, {}^{L}\Theta_{jj,\sigma}(\beta)\, \mathring{x}_j^{-1}$$

$$+ \sum_{j=1}^{3} \sum_{k=4}^{n} \overline{u_{ij}(\omega)}\, \overline{w_{ks}(\beta)}\, {}^{L}\Theta_{jk,\sigma}(\beta)\, \mathring{x}_j^{-1}$$

(III) $\left(\frac{\partial \overset{\circ}{x}}{\partial x}\right)_{is,q} = \frac{1}{2}\,\overline{u_{iq}(\omega)}\;\overline{w_{qs}(\beta)}$.

Proof: In order to determine K^{-1} we introduce an ansatz which contains matrices Θ inverse to Ξ's and undetermined matrices A and B.

(I) $\left(\frac{\partial w}{\partial x}\right)_{is,\tau} = \sum_{k=1}^{3}\sum_{j=1}^{3} \overline{u_{ik}(\omega)}\,\overline{w_{js}(\beta)}\,\frac{1}{2}\left[{}^{R}\Theta_{kj,\tau}(\omega) + A_{kj,\tau}\right]\overset{\circ}{x}{}^{-1}_{j}$

(II) $\left(\frac{\partial \beta}{\partial x}\right)_{is,\sigma} = \sum_{k=1}^{3}\sum_{j=1}^{3} \overline{u_{ij}(\omega)}\,\overline{w_{ks}(\beta)}\,\frac{1}{2}\left[{}^{L}\Theta_{jk,\sigma}(\beta) + B_{jk,\sigma}\right]\overset{\circ}{x}{}^{-1}_{j}$

$$+ \sum_{k=4}^{n}\sum_{j=1}^{3} \overline{u_{ij}(\omega)}\,\overline{w_{ks}(\beta)}\left[{}^{L}\Theta_{jk,\sigma}(\beta) + B_{jk,\sigma}\right]\overset{\circ}{x}{}^{-1}_{j}$$

(III) $\left(\frac{\partial \overset{\circ}{x}}{\partial x}\right)_{is,q} = \frac{1}{2}\,\overline{u_{iq}(\omega)}\;\overline{w_{qs}(\beta)}$.

Now we demand that $K K^{-1} = I$ and using (I.2.3) and (I.3.11) after a long but straightforward calculation we get for A and B the following expressions

$$A_{kj,\tau} = \frac{\overset{\circ}{x}{}^{2}_{j} + \overset{\circ}{x}{}^{2}_{k}}{\overset{\circ}{x}{}^{2}_{j} - \overset{\circ}{x}{}^{2}_{k}}\;{}^{R}\Theta_{kj,\tau}(\omega) \quad \text{for } k \neq j \;\; k,j = 1,2,3,$$

$$A_{kk,\tau} = -\,{}^{R}\Theta_{kk,\tau}(\omega) \quad \text{for } k = 1,2,3;$$

$$B_{jk,\sigma} = \frac{\overset{\circ}{x}{}^{2}_{j} + \overset{\circ}{x}{}^{2}_{k}}{\overset{\circ}{x}{}^{2}_{j} - \overset{\circ}{x}{}^{2}_{k}}\;{}^{L}\Theta_{jk,\sigma}(\beta) \quad \text{for } j \neq k \;\; j,k = 1,2,3,$$

$$B_{kk,\sigma} = 0 \quad \text{for } k = 1,2,3,$$

$$B_{jk,\sigma} = 0 \quad \text{for } j = 1,2,3 \;\; k = 4,5,\ldots,n. \;\square$$

(2) IV.2.1 Proposition: The Jacobi matrix for the transition

$(a_{(is)}, b_{(is)} \mid a := \mathrm{Re}x, b := \mathrm{Im}x) \to (\gamma_\mu, \mathring{x}_q, \beta_\sigma \mid \mu = 1,\dots,9 \; q = 1,2,3 \; \sigma = 1,\dots,6n-12) \equiv (\xi_\nu \mid \nu = 1,\dots,6n)$

is given through its matrix elements

$$\gamma_{\nu,(is)} : \longrightarrow \boxed{\left(\frac{\partial a}{\partial \xi}\right)_{\nu,(is)} \left(\frac{\partial b}{\partial \xi}\right)_{\nu,(is)}} = \boxed{\frac{\partial a_{(is)}}{\partial \xi_\nu} \quad \frac{\partial b_{(is)}}{\partial \xi_\nu}}$$

and can be written as the product

$$\gamma_{\nu,(is)} = \sum_{k=1}^{3} \sum_{t=1}^{n} K_{\nu,kt} \, C_{kt,(is)} \qquad \text{where}$$

$$C := \frac{1}{2} \begin{pmatrix} I_{3n} & -i I_{3n} \\ I_{3n} & i I_{3n} \end{pmatrix}$$

and

$$K_{\nu,is} : \to \boxed{\left(\frac{\partial x}{\partial \xi}\right)_{\nu,is} \left(\frac{\partial \bar{x}}{\partial \xi}\right)_{\nu,is}} = \boxed{\frac{\partial x_{is}}{\partial \xi_\nu} \quad \frac{\partial \bar{x}_{is}}{\partial \xi_\nu}}$$

(I) $$\left(\frac{\partial x}{\partial \delta}\right)_{\mu,is} = \sum_{q=1}^{3} \sum_{m=1}^{3} u_{im}(\delta) \, {}^{R}\mathcal{H}_{\mu,m_q}(\delta) \, \mathring{x}_q \, w_{qs}(\beta),$$

$$\left(\frac{\partial \bar{x}}{\partial \delta}\right)_{\mu,is} = \overline{\left(\frac{\partial x}{\partial \delta}\right)_{\mu,is}} \; ;$$

(II) $$\left(\frac{\partial x}{\partial \beta}\right)_{\sigma,is} = \sum_{q=1}^{3} \sum_{t=1}^{n} u_{iq}(\delta) \, \mathring{x}_q \, {}^{L}\mathcal{H}_{\sigma,qt}(\beta) \, w_{ts}(\beta),$$

$$\left(\frac{\partial \bar{x}}{\partial \beta}\right)_{\sigma,is} = \overline{\left(\frac{\partial x}{\partial \beta}\right)_{\sigma,is}} \; ;$$

(III) $$\left(\frac{\partial x}{\partial \mathring{x}}\right)_{q,is} = u_{iq}(\delta) \, w_{qs}(\beta),$$

$$\left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{q,is} = \overline{\left(\frac{\partial x}{\partial \mathring{x}}\right)_{q,is}} .$$

If one studies the action on analytic functions F(x) then the second part of the generators ${}^L\tau_{qi}(\bar{x})$ gives a contribution which is identically equal to zero.

(1) IV.3.2 Proposition: The left action generators of $\mathcal{U}(3)$ acting on the orbit $\mathcal{U}(3)\times U(n)/\mathcal{A}_3\times U(n-3)$, expressed through the new ω and β orbit coordinates are

$$ {}^L\chi_{iq} = -\sum_{\tau=1}^{6} {}^L\Theta_{iq,\tau}(\omega)\frac{\partial}{\partial\omega_\tau} - \sum_{\sigma=1}^{6n-9}\sum_{\ell=1}^{3} u_{q\ell}(\omega)\,\overline{u_{i\ell}(\omega)}\,{}^L\Theta_{\ell\ell,\sigma}(\beta)\frac{\partial}{\partial\beta_\sigma} \qquad i,q=1,2,3 $$

where the ω's are the parameters of the left coset space $\mathcal{U}(3)/\mathcal{A}_3$, and the β's are the parameters of the right coset space $U(n)/U(n-3)$.

Proof:

$$ \begin{aligned} {}^L\chi_{iq} = -\sum_{s=1}^{n}\Bigg\{ &\sum_{\tau=1}^{6}\left[x_{qs}\frac{\partial\omega_\tau}{\partial x_{is}} - \bar{x}_{is}\frac{\partial\omega_\tau}{\partial\bar{x}_{qs}}\right]\frac{\partial}{\partial\omega_\tau} \\ &+\sum_{p=1}^{3}\left[x_{qs}\frac{\partial\mathring{x}_p}{\partial x_{is}} - \bar{x}_{is}\frac{\partial\mathring{x}_p}{\partial\bar{x}_{qs}}\right]\frac{\partial}{\partial\mathring{x}_p} \\ &+\sum_{\sigma=1}^{6n-9}\left[x_{qs}\frac{\partial\beta_\sigma}{\partial x_{is}} - \bar{x}_{is}\frac{\partial\beta_\sigma}{\partial\bar{x}_{qs}}\right]\frac{\partial}{\partial\beta_\sigma}\Bigg\} \end{aligned} $$

Using explicit expressions for K^{-1} given in ((1) IV.2.2) and using relations (I.1.11) and (I.3.13), after a long but straightforward calculation, one gets the result. □

IV.3.3 Definition: Here we introduce the U(n) right action generators as differential operators acting on functions $F(x,\bar{x})$ where $x\in\mathbb{C}^{3n}$

$$ \begin{aligned} F'(x,\bar{x}) &:= F(x',\bar{x}') \\ &= F(x\tilde{g},\ \bar{x}\bar{\tilde{g}}) \ , \quad \tilde{g}(\tilde{\gamma})\in U(n). \end{aligned} $$

We use the exponential parametrization with $\tilde{\gamma}_{iq}$ being complex parameters $\overline{\tilde{\gamma}_{iq}} = -\tilde{\gamma}_{qi}$

$$\tilde{q}_{ie} = (\exp\tilde{\gamma})_{ie} = \sum_{m=0}^{\infty} \frac{1}{m!} (\tilde{\gamma}^m)_{ie} .$$

The right action on x with the element $\tilde{q}$ in the neighborhood of the identity is given by

$$x'_{it} = \sum_{s=1}^{n} x_{is}\, \tilde{q}_{st}$$
$$= x_{it} + \sum_{s=1}^{n} x_{is}\, \tilde{\gamma}_{st} + \dots$$

hence

$$F'(x,\bar{x}) \approx F(x,\bar{x}) + \sum_{s,t=1}^{n} \tilde{\gamma}_{st}\, {}^{R}X_{st}(x,\bar{x})\, F(x,\bar{x})$$

where

$${}^{R}X_{st}(x,\bar{x}) := {}^{R}C_{st}(x) - {}^{R}C_{ts}(\bar{x})$$
$$= {}^{R}C_{st}(x) - \overline{{}^{R}C_{ts}(x)}$$

and $${}^{R}C_{st}(x) := \sum_{i=1}^{3} x_{is} \frac{\partial}{\partial x_{it}} .$$

Those operators fulfil following commutation relations

$$\left[{}^{R}X_{st}, {}^{R}X_{vw} \right] = \delta_{tv}\, {}^{R}X_{sw} - \delta_{sw}\, {}^{R}X_{vt} .$$

One can come to the same expression for ${}^{R}X_{st}(x,\bar{x})$ using the Poisson generators of the group U(n) introduced in (IV.1.7) and fundamental Poisson brackets introduced in (IV.1.4)

$$\left\{ C_{st}, F \right\}(x,\bar{x}) = \sum_{i=1}^{3} \left(x_{is} \frac{\partial}{\partial x_{it}} - \overline{x_{it}} \frac{\partial}{\partial \overline{x_{is}}} \right) F(x,\bar{x})$$

$$=: {}^{R}X_{st}\, F(x,\bar{x}).$$

(1) IV.3.4 Proposition: The right action generators of U(n) acting on the orbit $\mathcal{U}(3)\times U(n) / \mathcal{A}_3 \times U(n-3)$, expressed through the new ω and β orbit coordinates are

$${}^{R}X_{st} = \sum_{\sigma=1}^{6n-9} {}^{R}\Theta_{st,\sigma}(\beta) \frac{\partial}{\partial \beta_\sigma} , \qquad s,t = 1,2,\dots,n$$

where the β's are the parameters of the right coset space U(n)/U(n-3).

Proof:

$$
{}^{R}\chi_{st} = \sum_{i=1}^{3} \Bigg\{ \sum_{\sigma=1}^{6} \Big[x_{is} \frac{\partial \omega_\sigma}{\partial x_{it}} - \overline{x}_{it} \frac{\partial \omega_\sigma}{\partial \overline{x}_{is}} \Big] \frac{\partial}{\partial \omega_\sigma}
$$

$$
+ \sum_{p=1}^{3} \Big[x_{is} \frac{\partial \mathring{x}_p}{\partial x_{it}} - \overline{x}_{it} \frac{\partial \mathring{x}_p}{\partial \overline{x}_{is}} \Big] \frac{\partial}{\partial \mathring{x}_p}
$$

$$
+ \sum_{\sigma=1}^{6n-9} \Big[x_{is} \frac{\partial \beta_\sigma}{\partial x_{it}} - \overline{x}_{it} \frac{\partial \beta_\sigma}{\partial \overline{x}_{is}} \Big] \frac{\partial}{\partial \beta_\sigma} \Bigg\} .
$$

To get the result, again we use the expression for K^{-1} given in ((1) IV.2.2) and also relations (I.1.11) and (I.3.13). □

The results that we obtain in ((1) IV.3.2) and ((1) IV.3.4) one should expect from the following global considerations for the left action:

$$
\begin{aligned}
\tilde{u}^{+}(\gamma) x &= \tilde{u}^{+}(\gamma)\, u(\omega) \mathring{x}\, w(\beta) \\
&= \tilde{u}^{+}(\gamma)\, u(\omega) \mathring{x}\, a_3'(\beta)\, w'(\beta) \\
&= u[w'(\gamma,\omega)]\, a_3(\gamma,\omega) \mathring{x}\, a_3'(\beta)\, w'(\beta) \\
&= u[w'(\gamma,\omega)] \mathring{x}\, a_3'[\beta'(\gamma,\omega,\beta)]\, w'(\beta)
\end{aligned}
$$

where $\tilde{u} \in \mathcal{U}(3)$ (acting group), $u \in \mathcal{U}(3)/A_3$, $a_3 \in A_3 < \mathcal{U}(3)$ $a_3' \in A_3 < U(n)$, $w \in U(n)/U(n-3)$, $w' \in U(n)/A_3 \times U(n-3)$; for the U(n) right action:

$$
\begin{aligned}
x \hat{g}(\gamma) &= u(\omega) \mathring{x}\, w(\beta)\, \tilde{g}(\gamma) \\
&= u(\omega) \mathring{x}\, a_3'(\beta,\gamma)\, w'(\beta,\gamma)
\end{aligned}
$$

where $\hat{g} \in U(n)$ (acting group), $w' \in U(n)/A_3 \times U(n-3)$, $a_3' \in A_3 < U(n)$.

(2) IV.3.2 Proposition: The left action generators of $\mathcal{U}(3)$ acting on the orbit $\mathcal{U}(3) \times U(n)/A_3 \times U(n-3)$, expressed through the new γ and β orbit coordinates are

$$
{}^{L}\chi_{iq} = -\sum_{\mu=1}^{9} {}^{L}\Theta_{iq,\mu}(\gamma) \frac{\partial}{\partial \gamma_\mu} , \qquad i,q = 1,2,3
$$

where the γ's are the parameters of the group $\mathcal{U}(3)$.

(2) IV.3.4 Proposition: The right action generators of U(n) acting on the orbit $\mathcal{U}(3)\times U(n)/A_3\times U(n-3)$ expressed through the new γ and β orbit coordinates are

$$ {}^R X_{st} = \sum_{\sigma=1}^{6n-12} {}^R\Theta_{st,\sigma}(\beta)\frac{\partial}{\partial\beta_\sigma} + \sum_{\mu=1}^{9}\sum_{\ell=1}^{3} w_{\ell s}(\beta)\,\overline{w_{\ell t}(\beta)}\,{}^R\Theta_{\ell\ell,\mu}(\gamma)\frac{\partial}{\partial\gamma_\mu} $$

where the γ's are the parameters of the group $\mathcal{U}(3)$, and β's are the parameters of the right coset space $U(n)/A_3\times U(n-3)$.

IV.4. The Bargmann measure expressed through the new coordinates

The measure on the Bargmann space of square integrable analytic functions F(x) is given by the expression[49]

$$ d\mu(x) = \pi^{-3n}\prod_{j=1}^{3n}\exp(-\bar{x}_j x_j)\, d\,\mathrm{Re}\,x_j\, d\,\mathrm{Im}\,x_j $$

and for $x_j = a_j + i b_j$, $j \leftrightarrow is$ $i=1,2,3$ $s=1,2,\dots,n$ $j=1,2,\dots,3n$

$$ d\mu(x) = \pi^{-3n}\prod_{j=1}^{3n}\exp(-\bar{x}_j x_j)\, da_j\, db_j . $$

If one goes over to the coordinates on the orbit $\mathcal{U}(3)\times U(n)/A_3\times U(n-3$ in the case

$$ (1)\ (a_{(is)}, b_{(is)}|_{i=1,2,3\ s=1,2,\dots,n}) \to (\omega_\alpha, \mathring{x}_q, \beta_\sigma|_{\alpha=1,2,\dots,6\ q=1,2,3\ \sigma=1,2,\dots,6n-} $$

then the expression for the measure is

$$ d\mu = \pi^{-3n}\exp\Big(-\sum_{\ell=1}^{3}\mathring{x}_\ell^2\Big)\det\gamma\prod_{\alpha=1}^{6} d\omega_\alpha \prod_{q=1}^{3} d\mathring{x}_q \prod_{\sigma=1}^{6n-9} d\beta_\sigma . $$

In order to determine this measure, one should calculate the Jacobian. The Jacobi matrix for the transition $(a_{(is)}, b_{(is)}) \to (\xi_\nu)$ is given by ((1) IV.2.1), $M_\gamma = KC$, where

$$ C = \frac{1}{2}\begin{pmatrix} I_{3n} & -iI_{3n} \\ I_{3n} & iI_{3n}\end{pmatrix} \quad\text{and}\quad K_{\nu,is}: \to \boxed{\begin{matrix}\frac{\partial x_{is}}{\partial\xi_\nu} & \frac{\partial\bar{x}_{is}}{\partial\xi_\nu}\end{matrix}} $$

and where ξ_ν are some real coordinates, $\nu=1,2,\dots,6n$. We write

$$ \det M_\gamma = [\det M_\gamma^t M_\gamma]^{\frac{1}{2}} \text{ and} $$

$$ M_\gamma^t M_\gamma = M_\gamma M_\gamma^+ $$

$$ = \frac{1}{2} K K^+ $$

so $\det M_\gamma^t M_\gamma = 2^{-6n}\det KK^+$.

The expression for the Bargmann measure is now

$$d\mu = (2\pi)^{-3n} \exp\Big(-\sum_{\ell=1}^{3} \mathring{x}_\ell^2\Big) \left(\det K K^+\right)^{\frac{1}{2}} \prod d\omega \prod d\mathring{x} \prod d\beta, \qquad \text{where}$$

$$K K^+ = \boxed{\left(\frac{\partial x}{\partial \xi}\right) \left(\frac{\partial \bar{x}}{\partial \xi}\right)} \boxed{\begin{matrix} {}^t\left(\frac{\partial \bar{x}}{\partial \xi}\right) \\ {}^t\left(\frac{\partial x}{\partial \xi}\right) \end{matrix}} \quad , \; {}^t\left(\frac{\partial x}{\partial \xi}\right)_{is,\nu} := \left(\frac{\partial x}{\partial \xi}\right)_{\nu,is}$$

$$(KK^+)_{\tau\tau'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \omega}\right)_{\tau,is} {}^t\left(\frac{\partial \bar{x}}{\partial \omega}\right)_{is,\tau'} + \left(\frac{\partial \bar{x}}{\partial \omega}\right)_{\tau,is} {}^t\left(\frac{\partial x}{\partial \omega}\right)_{is,\tau'} \Big]$$

$$(KK^+)_{\tau q'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \omega}\right)_{\tau,is} {}^t\left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{is,q'} + \left(\frac{\partial \bar{x}}{\partial \omega}\right)_{\tau,is} {}^t\left(\frac{\partial x}{\partial \mathring{x}}\right)_{is,q'} \Big]$$

$$(KK^+)_{\tau\sigma'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \omega}\right)_{\tau,is} {}^t\left(\frac{\partial \bar{x}}{\partial \beta}\right)_{is,\sigma'} + \left(\frac{\partial \bar{x}}{\partial \omega}\right)_{\tau,is} {}^t\left(\frac{\partial x}{\partial \beta}\right)_{is,\sigma'} \Big]$$

$$(KK^+)_{q\tau'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \mathring{x}}\right)_{q,is} {}^t\left(\frac{\partial \bar{x}}{\partial \omega}\right)_{is,\tau'} + \left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{q,is} {}^t\left(\frac{\partial x}{\partial \omega}\right)_{is,\tau'} \Big]$$

$$(KK^+)_{qq'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \mathring{x}}\right)_{q,is} {}^t\left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{is,q'} + \left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{q,is} {}^t\left(\frac{\partial x}{\partial \mathring{x}}\right)_{is,q'} \Big]$$

$$(KK^+)_{q\sigma'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \mathring{x}}\right)_{q,is} {}^t\left(\frac{\partial \bar{x}}{\partial \beta}\right)_{is,\sigma'} + \left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{q,is} {}^t\left(\frac{\partial x}{\partial \beta}\right)_{is,\sigma'} \Big]$$

$$(KK^+)_{\sigma\tau'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \beta}\right)_{\sigma,is} {}^t\left(\frac{\partial \bar{x}}{\partial \omega}\right)_{is,\tau'} + \left(\frac{\partial \bar{x}}{\partial \beta}\right)_{\sigma,is} {}^t\left(\frac{\partial x}{\partial \omega}\right)_{is,\tau'} \Big]$$

$$(KK^+)_{\sigma q'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \beta}\right)_{\sigma,is} {}^t\left(\frac{\partial \bar{x}}{\partial \mathring{x}}\right)_{is,q'} + \left(\frac{\partial \bar{x}}{\partial \beta}\right)_{\sigma,is} {}^t\left(\frac{\partial x}{\partial \mathring{x}}\right)_{is,q'} \Big]$$

$$(KK^+)_{\sigma\sigma'} = \sum_{i=1}^{3} \sum_{s=1}^{n} \Big[\left(\frac{\partial x}{\partial \beta}\right)_{\sigma,is} {}^t\left(\frac{\partial \bar{x}}{\partial \beta}\right)_{is,\sigma'} + \left(\frac{\partial \bar{x}}{\partial \beta}\right)_{\sigma,is} {}^t\left(\frac{\partial x}{\partial \beta}\right)_{is,\sigma'} \Big] .$$

Using the explicit expressions for the matrix elements of K in ((1) IV.2.1) and hermiticity of KK^+ we find

$$(KK^+)_{\tau\tau'} = 2 \sum_{m=1}^{3} \sum_{\ell=1}^{3} {}^{R2}\mathcal{F}_{\tau,m\ell}(\omega) \tfrac{1}{2} \Big[(\mathring{x}_\ell)^2 + (\mathring{x}_m)^2 \Big] \, {}^t \overline{\left[{}^{R2}\mathcal{F} \right]_{m\ell,\tau'}(\omega)}$$

$$(KK^+)_{\tau q'} = 0$$

$$(KK^+)_{\tau\sigma'} = 2 \sum_{m=1}^{3} \sum_{\ell=1}^{3} {}^{R2}\mathcal{F}_{\tau,m\ell}(\omega) \, \mathring{x}_\ell \, \mathring{x}_m \, {}^t \overline{\left[{}^{L2}\mathcal{F} \right]_{m\ell,\sigma'}(\beta)}$$

$$(KK^+)_{q\tau'} = 0$$

$$(KK^+)_{qq'} = 2\,\delta_{qq'}$$

$$(KK^+)_{q\sigma'} = 0$$

$$(KK^+)_{\sigma\tau'} = 2\sum_{m=1}^{3}\sum_{\ell=1}^{3} {}^{L}\mathcal{F}_{\sigma,m\ell}(\beta)\,\mathring{x}_\ell\,\mathring{x}_m\,{}^{t}\overline{[{}^{R}\mathcal{F}]_{m\ell,\tau'}(\omega)}$$

$$(KK^+)_{\sigma q'} = 0$$

$$(KK^+)_{\sigma\sigma'} = 2\sum_{m=1}^{3}\sum_{\ell=1}^{n}(\mathring{x}_m)^2\,\frac{1}{2}\Big[{}^{L}\mathcal{F}_{\sigma,m\ell}(\beta)\,{}^{t}\overline{[{}^{L}\mathcal{F}]_{m\ell,\sigma'}(\beta)} + {}^{L}\overline{\mathcal{F}_{\sigma,m\ell}(\beta)}\,{}^{t}[{}^{L}\mathcal{F}]_{m\ell,\sigma'}$$

Rearranging some rows and columns and using the block structure of KK^+, we get the following expression for $\det(KK^+)$

$$\det(KK^+) = 2^{6n}\det\left[(\tilde{K}\tilde{K}^+)_{\sigma\sigma'}\right]\det\left[(\tilde{K}\tilde{K}^+)_{q\tau'} - \sum_{\sigma}\sum_{\sigma'}(\tilde{K}\tilde{K}^+)_{q\sigma}\,M^{-1}_{\sigma\sigma'}\,(\tilde{K}\tilde{K}^+)_{\sigma'\tau'}\right]$$

where $\tilde{K} := \frac{1}{\sqrt{2}}K$ and $M^{-1}_{\sigma\sigma'}$ means the inverse of the matrix block $(\tilde{K}\tilde{K}^+)_{\sigma\sigma'} =: M_{\sigma\sigma'}$. The matrix block $(\tilde{K}\tilde{K}^+)_{\sigma\sigma'}$ can be written as a product of quadratic (6n-9)x(6n-9) matrices

$$(\tilde{K}\tilde{K}^+)_{\sigma\sigma'} = \sum_{\sigma''}\sum_{m\ell}\sum_{kw}\sum_{\sigma'''} {}^{L}\mathcal{F}_{\sigma\sigma''}(\beta)\,P_{\sigma'',m\ell}\,Q_{m\ell,kw}\,{}^{t}\overline{P_{kw,\sigma'''}}\,{}^{t}\overline{[{}^{L}\mathcal{F}]_{\sigma'''\sigma'}(\beta)}$$

where P is given in ((1) IV.1.16) and Q is a diagonal matrix with non zero elements:

$$Q_{m\ell,kw} = \begin{cases} \frac{1}{2}\mathring{x}_m^2\,\delta_{m\ell,kw} \text{ where } \delta_{m\ell,kw} \equiv \delta_{mk}\delta_{\ell w} \text{ for } m,k=1,2,3 \;\; \ell,w=4,5,\dots,n \\ \frac{1}{2}\mathring{x}_\ell^2\,\delta_{m\ell,kw} \text{ for } m,k=4,5,\dots,n \;\; \ell,w=1,2,3 \\ \frac{1}{2}(\mathring{x}_m^2+\mathring{x}_\ell^2)\,\delta_{m\ell,kw} \text{ for } m,\ell,k,w=1,2,3. \end{cases}$$

so, $\det\left[(\tilde{K}\tilde{K}^+)_{\sigma\sigma'}\right] = \left|\det {}^{L}\mathcal{F}_{\sigma\sigma''}(\beta)\right|^2 \det Q\,\left|\det P_{\sigma'',m\ell}\right|^2$ where $\left|\det P_{\sigma'',m\ell}\right| = 2^{3n-6}$ and $\det Q = 2^{2(3n-6)}\,\mathring{x}_m^{2(2n-5)}\prod_{m\ell}^{3}(\mathring{x}_m^2+\mathring{x}_\ell^2)^2$.

The matrix block $(\tilde{K}\tilde{K}^{+})_{\sigma\sigma'}$, can be easily inverted using the relation (I.2.3) and the fact that Q is a diagonal matrix. After some calculation one gets that

$$(\tilde{K}\tilde{K}^{+})_{\tau\tau'} - \sum_{\sigma}\sum_{\sigma'}(\tilde{K}\tilde{K}^{+})_{\tau\sigma}\, M^{-1}_{\sigma\sigma'}\,(\tilde{K}\tilde{K}^{+})_{\sigma'\tau'} =$$

$$= \sum_{\substack{n=1\\(n\neq\ell)}}^{3}\sum_{\ell=1}^{3}\sum_{k=1}^{3}\sum_{w=1}^{3} {}^{R}2\mathcal{F}_{\tau,n\ell}(\omega)\frac{1}{2}\frac{(\mathring{x}_n^2-\mathring{x}_\ell^2)^2}{(\mathring{x}_n^2+\mathring{x}_\ell^2)}\delta_{n\ell,kw}\,\overline{{}^{t}({}^{R}2\mathcal{F})_{kw,\tau}(\omega)}.$$

For the last expression it is easy to find the determinant, and finally we get

$$d\mu = \pi^{-3n}\det {}^{R}2\mathcal{F}_{\tau\tau'}(\omega)\prod_{\tau=1}^{6} d\omega_\tau \Big\{\exp\Big(-\sum_{\ell=1}^{3}\mathring{x}_\ell^2\Big)\prod_{m=1}^{3}\mathring{x}_m^{2n-5}\times$$

$$\times\prod_{\substack{n=1\\(n<\ell)}}^{3}\prod_{\ell=1}^{3}(\mathring{x}_n^2-\mathring{x}_\ell^2)^2\prod_{q=1}^{3} d\mathring{x}_q\Big\}\det {}^{L}2\mathcal{F}_{\sigma\sigma'}(\beta)\prod_{\sigma=1}^{6n-9} d\beta_\sigma, \quad \text{or}$$

$$d\mu = \pi^{-3n}\, dm(\omega)\Big\{\exp\Big(-\sum_{\ell=1}^{3}\mathring{x}_\ell^2\Big)\prod_{m=1}^{3}\mathring{x}_m^{2n-5}\prod_{\substack{n=1\\(n<\ell)}}^{3}\prod_{\ell=1}^{3}(\mathring{x}_n^2-\mathring{x}_\ell^2)^2\prod_{q=1}^{3} d\mathring{x}_q\Big\}\, dm(\beta)$$

where we made use of (I.2.4), which applies to the unitary groups because they have the property that $\det d(\delta)=1$.[38]
If one would like to construct a state as a function of the new variables, it is not sufficient to demand that it be square integrable with respect to the new measure. In addition one has to require that it be analytic when expressed in the variables x and $\bar{x}$.
If one goes over to the coordinates on the orbit $\mathcal{U}(3)\times U(n)/A_3\times U(n-3)$ in the case (2) $(a_{(is)},b_{(is)}\,|\,i=1,2,3\ s=1,\dots,n)\to(\delta_\mu,\mathring{x}_q,\beta_\sigma\,|\,\mu=1,\dots,9\ \ q=1,2,3\ \ \sigma=1,2,\dots,6n-12)$ then the expression for the measure on the Bargmann space is

$$d\mu(x) = \pi^{-3n}\ \ dm(\delta)\Big\{\exp\Big(-\sum_{j=1}^{3}\mathring{x}_j^2\Big)\prod_{j=1}^{3}(\mathring{x}_j)^{2n-5}\prod_{\substack{j,m\\(j<m)}}^{3}(\mathring{x}_j^2-\mathring{x}_m^2)^2\prod_{q=1}^{3} d\mathring{x}_q\Big\}\, dm(\beta)$$

where $dm(\delta)=\det {}^{R}\mathcal{F}(\delta)\prod_{\delta=1}^{9} d\delta_\mu$ and $dm(\beta)=\det {}^{L}2\mathcal{F}(\beta)\prod_{\sigma=1}^{6n-12} d\beta_\sigma$

are the measures on the $\mathcal{U}(3)$-group space and $U(n)/A_3\times U(n-3)$-right coset space, respectively.

V. The $gl(3,\mathbb{C})\times U(n)$ scheme

V.0 Introduction

In chapter V we choose as the collective group the complex general linear group $gl(3,\mathbb{C})$. The group $gl(3,\mathbb{C})$ has the same generators as the group $u(3)$ considered in chapter IV, but its Lie algebra is obtained by allowing for complex linear combinations of these generators. When acting in phase space, $gl(3,\mathbb{C})$ still commutes with U(n). In (V.1) the phase space is decomposed into orbits under $gl(3,\mathbb{C})\times U(n)$. Various orbits of the unitary scheme studied in ref. 35 and chapter IV are now combined into a single new orbit. In the $gl(3,\mathbb{C})\times U(n)$ scheme both unitary schemes are also included, in chapter IV referred to as (1) and (2). In (V.2) the Jacobi matrix and its inverse are obtained between the new and old coordinates of phase space. The generators of the acting groups $u(3)$ and U(n) are transformed to new coordinates in (V.3). The new form of state space is considered in (V.4) and (V.5). In (V.4), the measure of Bargmann space is transformed to the new coordinates. The coordinates introduced in the $gl(3,\mathbb{C})\times U(n)$ scheme are especially handy if one wants to construct the basis in the Bargmann space. A complete orhtonormal basis in the new coordintaes is constructed in (V.5). It consists of sums of products of collective and intrinsic states. The collective states are polynomials in a new Bargmann space of analytic functions in nine complex variables. The intrinsic states are matrix elements of irreducible representations of U(n) on a coset space of this group. An U(n)-invariant collective Hamiltonian can be analyzed on the collective state space of Bargmann type.

V.1. Collective and intrinsic coordinates, orbits

The action of groups $gl(3,\mathbb{C})$ and U(n) is, in an analogous fashion as in the unitary scheme, defined on the complex realization of phase space $\mathcal{P}$, whose elements are pairs $(x_{is}, \overline{x_{is}})$

$$x_{is} = \frac{1}{\sqrt{2}}\left(\xi_{(is)} - i\,\pi_{(is)}\right) \equiv a_{(is)} + ib_{(is)}$$

$$\overline{x_{is}} = \frac{1}{\sqrt{2}}\left(\xi_{(is)} + i\,\pi_{(is)}\right).$$

$\xi_{(is)}$ are the real cartesian coordinates, $a \equiv \frac{1}{\sqrt{2}}\xi$, and $\pi_{(is)}$ are the momenta, $b \equiv -\frac{1}{\sqrt{2}}\pi$, i = 1,2,3 are the configuration space indices and s = 1,2,...n are the Jacobi vector indices. Two latin indices in a bracket mean the parametrization of real coordinates in the same sense as in (I.3) and (IV.2). Let us introduce the action on $\mathcal{P}$.

V.1.1 Definition:

$$\left[gl(3,\mathbb{C}) \times U(n)\right] \times \mathcal{P} \to \begin{pmatrix} \tilde{x}^{\prime-1} & 0 \\ 0 & \overline{\tilde{x}^{\prime-1}} \end{pmatrix} \begin{pmatrix} x & 0 \\ 0 & \bar{x} \end{pmatrix} \begin{pmatrix} \tilde{q} & 0 \\ 0 & \overline{\tilde{q}} \end{pmatrix}$$

$$= \begin{pmatrix} \tilde{x}^{\prime-1} x \tilde{q} & 0 \\ 0 & \overline{\tilde{x}^{\prime-1} x \tilde{q}} \end{pmatrix}$$

where $\tilde{x}' \in gl(3,\mathbb{C})$, $x = (x_{is}) \in \mathcal{M} \simeq \mathbb{C}^{3n}$ and $\tilde{q} \in U(n)$.

Once the orbit analysis is implemented according to $x \to \tilde{x}^{\prime-1} x \tilde{q}$ on $\mathcal{M}$, it can be immediately extended to the full phase space $\mathcal{P}$. In the orbit analysis we proceed in a direct way. We choose the representative point to be $x_{is} = \delta_{is}$, i = 1,2,3 s = 1,2,...,n and find its stability group by demanding that

$$\sum_{j=1}^{3} \sum_{t=1}^{n} \tilde{x}^{\prime-1}_{ij}\, \delta_{jt}\, \tilde{q}_{ts} = \delta_{is}.$$

The stability group is isomorphic to U(3)xU(n-3), where U(n-3) is a subgroup of U(n) whereas U(3) can be chosen to be either a subgroup of $gl(3,\mathbb{C})$ or of U(n). We choose U(3) to be a subgroup of U(n). Then the orbit for the representative point given above is the coset space.

$$g\mathcal{L}(3,\mathbb{C}) \times \left(U(n) \,|\, U(3) \times U(n-3) \right)$$

and is parametrized by $\{ 18 + (n^2 - 9 - (n-3)^2 \} = 6n$ real parameters. In order to show that the orbit discussed so far is only one under the $g\mathcal{L}(3,\mathbb{C}) \times U(n)$ action, we now make use of the unitary scheme. From this scheme we know that $g\mathcal{L}(3,\mathbb{C})$ may be decomposed as

$$g\mathcal{L}(3,\mathbb{C}) = U \left\{ (\mathcal{U}(3) / \mathcal{H}_3) \times \mathring{x} \times U(3) \right\}$$

where all the different stability points $\mathring{x}$ and corresponding "parts" of the stability group $\mathcal{H}_3$ appear (see Table 1 in ref. 35). Upon inserting this decomposition of $g\mathcal{L}(3,\mathbb{C})$ into the coset space given above it can be seen that this coset space contains all the orbits of the unitary scheme for $n \geq 3$. Since in the unitary scheme an exhaustive decomposition of the manifold $\mathcal{M}$ was obtained, the new coset space clearly exhausts $\mathcal{M}$, except for points where both $\mathring{x}$ and x' are non-invertible.

V.1.2 Proposition: Under the action of $g\mathcal{L}(3,\mathbb{C}) \times U(n)$, the manifold $\mathcal{M}$ becomes a single orbit equivalent to the coset space

$$g\mathcal{L}(3,\mathbb{C}) \times \left(U(n) \,|\, U(3) \times U(n-3) \right).$$

We write for $x \in \mathcal{M}$

$$x_{is} = \sum_{j=1}^{3} x'_{ij} \, w_{js}(\alpha)$$

where $x' \in g\mathcal{L}(3,\mathbb{C})$ and is parametrized by complex parameters

$$\left(x'_{ij}, \overline{x'_{ij}} \,|\, i,j = 1,2,3 \right) \qquad \text{and} \qquad w(\alpha) \in U(n) \,|\, \left(U(3) \times U(n-3) \right),$$

a right coset space. If the "old" set of coordinates in phase space were $\left(x_{is}, \overline{x_{is}} \,|\, i = 1,2,3 \;\; s = 1,2,\ldots,n \right)$, the "new" coordinates can be chosen as $\left(x'_{ij}, \overline{x'_{ij}}, \alpha_\sigma \,|\, i,j = 1,2,3 \;\; \sigma = 1,2,\ldots,6n-18 \right)$. The 18 coordinates $(x'_{ij}, \overline{x'_{ij}})$ we call collective, and they represent an enlarged set of collective coordinates compared to those in the unitary scheme in both cases (1) and (2). The 6n-18 coordinates

(α_σ) we call intrinsic, and they are a reduced set of intrinsic coordinates compared with the corresponding (β_σ) in the unitary scheme.

Within the set of new coordinates with respect to the $\mathfrak{gl}(3,\mathbb{C}) \times U(n)$ scheme, we are able not only to separate the Bargmann measure into a purely collective and purely intrinsic part, but also to define the complete set of orthonormal basis functions factorized in a corresponding way (see (V.5)).

V.1.3 Definition: The P matrix for U(n) in the $\mathfrak{gl}(3,\mathbb{C}) \times U(n)$ scheme is

$P_{\mu,st} =$

	$U(3)$		$U(n-3)$		$U(n) \mid U(3) \times U(n-3)$
	$s=t=1,2,3$	$s\neq t$ $s,t\leq 3$	$s=\pm=4,5,\ldots,n$ $s\neq t$ $s,t\geq 4$		$s\geq 4$ $t\leq 3$, $t\geq 4$ $s\leq 3$
$\varkappa$	$i\,I_3$	0	0	0	0
	0	$\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \times I_3$	0	0	0
	0	0	$i\,I_{n-3}$	0	0
	0	0	0	$\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \times I_{\frac{(n-3)(n-4)}{2}}$	0
σ	0	0	0	0	$\begin{pmatrix} 1 & -1 \\ i & i \end{pmatrix} \times I_{3n-9}$

V.2. The Jacobi matrix and its inverse

In (V.3) we shall express the generators of the left action of $\mathcal{U}(3)$ and the right action of U(n) on the only existing orbit of the $\mathcal{GL}(3,\mathbb{C})\times U(n)$ scheme through the coordinates $(x', \bar{x}', \alpha)$. We also want (in (V.4)) to determine the Bargmann measure expressed through the new coordinates and to separate the intrinsic part dependent on α from the "enlarged" collective part depending on x' and $\bar{x}'$. One way to achieve that is through the Jacobi matrix. So, let us determine it.

V.2.1 Proposition: The Jacobi matrix for the transition

$$\left(a_{(is)}, b_{(is)} \,|\, a \equiv Re\, x, b = Im\, x,\ i=1,2,3\ \ s=1,2,\dots,n\right) \rightarrow$$

$$\rightarrow \left(a'_{(ik)}, b'_{(ik)}, \alpha_\sigma \,|\, a' \equiv Re\, x', b' \equiv Im\, x',\ \ i,k=1,2,3\ \ \sigma=1,2,\dots,6n-18\right)$$

$$\equiv \left(\xi_\nu \,|\, \nu=1,2,\dots,6n\right)$$

is given through its matrix elements

$$M_{\nu,(is)} : \rightarrow \boxed{\left(\frac{\partial a}{\partial \xi}\right)_{\nu,(is)} \left(\frac{\partial b}{\partial \xi}\right)_{\nu,(is)}} \equiv \boxed{\frac{\partial a_{(is)}}{\partial \xi_\nu} \quad \frac{\partial b_{(is)}}{\partial \xi_\nu}}$$

and can be written as the product of three 6n x 6n matrices N, L and C,

$$M_{\nu,(is)} = \sum_{[\,]} \sum_{j=1}^{3} \sum_{t=1}^{n} N_{\nu,[\,]}\, L_{[\,],jt}\, C_{jt,(is)}$$

where both ν and (is) are indices of the real parameters (coordinates), compare (I.3), and $\sum_{[\,]}$ denotes a summation over both, indices of complex and real parameters $[\,] \equiv (\ell k, \ell k, \sigma \,|\, \ell,k=1,2,3\ \ \sigma=1,2,\dots,6$

The matrix N is constructed from the 3 x 3 blocks

$$\begin{pmatrix} 1 & 1 & 0 \\ i & -i & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

the matrix C in the same manner as in (IV.2.1) through

$$\frac{1}{2}\begin{pmatrix} 1 & -i \\ 1 & i \end{pmatrix}.$$

The matrix L is given in block form by

$$L_{[J],is;} \rightarrow \boxed{\begin{matrix} \left(\frac{\partial x}{\partial x'}\right)_{ek,is} & \left(\frac{\partial \bar{x}}{\partial x'}\right)_{ek,is} \\ \left(\frac{\partial x}{\partial \bar{x}'}\right)_{ek,is} & \left(\frac{\partial \bar{x}}{\partial \bar{x}'}\right)_{ek,is} \\ \left(\frac{\partial x}{\partial \alpha}\right)_{\sigma,is} & \left(\frac{\partial \bar{x}}{\partial \alpha}\right)_{\sigma,is} \end{matrix}} \equiv \boxed{\begin{matrix} \frac{\partial x_{is}}{\partial x'_{ek}} & \frac{\partial \bar{x}_{is}}{\partial x'_{ek}} \\ \frac{\partial x_{is}}{\partial \bar{x}'_{ek}} & \frac{\partial \bar{x}_{is}}{\partial \bar{x}'_{ek}} \\ \frac{\partial x_{is}}{\partial \alpha_{\sigma}} & \frac{\partial \bar{x}_{is}}{\partial \alpha_{\sigma}} \end{matrix}}$$

and the blocks of L have the explicit form

(I) $\left(\frac{\partial x}{\partial x'}\right)_{ek,is} = \delta_{ei} w_{ks}$, $\qquad \left(\frac{\partial \bar{x}}{\partial x'}\right)_{ek,is} = 0$;

(II) $\left(\frac{\partial x}{\partial \bar{x}'}\right)_{ek,is} = \left(\frac{\partial \bar{x}}{\partial x'}\right)_{ek,is}$, $\qquad \left(\frac{\partial \bar{x}}{\partial \bar{x}'}\right)_{ek,is} = \overline{\left(\frac{\partial x}{\partial x'}\right)_{ek,is}}$;

(III) $\left(\frac{\partial x}{\partial \alpha}\right)_{\sigma,is} = \sum_{v=1}^{3} \sum_{m=1}^{n} x'_{iv}\, {}^{L}T_{\sigma,vm}(\alpha)\, w_{ms}(\alpha)$, $\qquad \left(\frac{\partial \bar{x}}{\partial \alpha}\right)_{\sigma,is} = \overline{\left(\frac{\partial x}{\partial \alpha}\right)_{\sigma,is}}$.

Proof: For (I), (II) and (III) we differentiate the relation

$$x_{is} = \sum_{k=1}^{3} x'_{ik}\, w_{ks}(\alpha) \; ; \quad x' \in \mathcal{GL}(3,\mathbb{C}), \; w(\alpha) \in U(n) \,|\, U(3) \times U(n-3)$$

with respect to the new coordinates x', $\bar{x}'$ and α , respectively. To determine (III) we also use (I.3.12). As parameters for $\mathcal{GL}(3,\mathbb{C})$ we use the matrix elements of its fundamental representation

$\left(x'_{ik} \,|\, i,k = 1,2,3 \;\; x' \in \mathcal{GL}(3,\mathbb{C})\right)$. □

V.2.2 Proposition: The inverse Jacobi matrix M^{-1} is given in terms of N^{-1}, C^{-1} and L^{-1} where

$$(L^{-1})_{is,[\,]}: \rightarrow \quad \begin{array}{|ccc|} \hline \left(\frac{\partial x'}{\partial x}\right)_{is,\ell k} & \left(\frac{\partial \bar{x}'}{\partial x}\right)_{is,\ell k} & \left(\frac{\partial \alpha}{\partial x}\right)_{is,\sigma} \\ \left(\frac{\partial x'}{\partial \bar{x}}\right)_{is,\ell k} & \left(\frac{\partial \bar{x}'}{\partial \bar{x}}\right)_{is,\ell k} & \left(\frac{\partial \alpha}{\partial \bar{x}}\right)_{is,\sigma} \\ \hline \end{array}$$

$$\equiv \quad \begin{array}{|ccc|} \hline \frac{\partial x'_{\ell k}}{\partial x_{is}} & \frac{\partial \bar{x}'_{\ell k}}{\partial x_{is}} & \frac{\partial \alpha_\sigma}{\partial x_{is}} \\ \frac{\partial x'_{\ell k}}{\partial \bar{x}_{is}} & \frac{\partial \bar{x}'_{\ell k}}{\partial \bar{x}_{is}} & \frac{\partial \alpha_\sigma}{\partial \bar{x}_{is}} \\ \hline \end{array}$$

and the blocks of L^{-1} have the explicit form

(I) $$\left(\frac{\partial x'}{\partial x}\right)_{is,\ell k} = \delta_{i\ell}\,\overline{w_{ks}(\alpha)} + \sum_{z=1}^{3}\sum_{y=4}^{n}\sum_{p=1}^{3} \overline{w_{ys}(\alpha)}\, x'^{-1}_{zi}\, \overline{\ominus}_{zy,pk}(\alpha)\, x'_{\ell p},$$

$$\left(\frac{\partial \bar{x}'}{\partial \bar{x}}\right)_{is,\ell k} = \overline{\left(\frac{\partial x'}{\partial x}\right)_{is,\ell k}};$$

(II) $$\left(\frac{\partial \bar{x}'}{\partial x}\right)_{is,\ell k} = -\sum_{z=1}^{3}\sum_{y=4}^{n}\sum_{p=1}^{3} \overline{w_{ys}(\alpha)}\, x'^{-1}_{zi}\, \overline{\ominus_{yz,pk}(\alpha)}\, \overline{x'_{\ell p}},$$

$$\left(\frac{\partial x'}{\partial \bar{x}}\right)_{is,\ell k} = \overline{\left(\frac{\partial \bar{x}'}{\partial x}\right)_{is,\ell k}};$$

(III) $$\left(\frac{\partial \alpha}{\partial x}\right)_{is,\sigma} = \sum_{z=1}^{3}\sum_{y=4}^{n} \overline{w_{ys}(\alpha)}\, x'^{-1}_{zi}\, \overline{\ominus}_{zy,\sigma}(\alpha),$$

$$\left(\frac{\partial \alpha}{\partial \bar{x}}\right)_{is,\sigma} = \overline{\left(\frac{\partial \alpha}{\partial x}\right)_{is,\sigma}}.$$

Proof: It can be easily seen the the matrix L can be written as a product

$$L_{[\,],is} = \sum_{\{jm\}} A_{[\,],\{jm\}} \, \mathcal{D}_{\{jm\},is} \rightarrow$$

$A_{\ell k,jm}$	0	0	0
0	0	$\overline{A_{\ell k,jm}}$	0
$(\tilde{A}_1)_{\sigma,jm}$	$(\tilde{A}_2)_{\sigma,jm}$	$\overline{(\tilde{A}_1)_{\sigma,jm}}$	$(\tilde{A}_2)_{\sigma,jm}$

$D_{jm,is}$	0
0	$\overline{D_{jm,is}}$

where $A_{\ell k,jm} = {}^t x^{-1}_{\ell j} \, \delta_{km}$ for $\ell,k,j,m = 1,2,3$; $(\tilde{A}_1)_{\sigma,jm} = {}^L\mathcal{F}_{\sigma,jm}(\alpha)$ for $\sigma = 1,2,\ldots,6n-18$ $j,m = 1,2,3$; $(\tilde{A}_2)_{\sigma,jm} = {}^L\mathcal{F}_{\sigma,jm}(\omega)$ for $\sigma = 1,2,\ldots,6n-18$ $j = 1,2,3$ $m = 4,5,\ldots,n$ and $D_{jm,is} = {}^t x'_{ji} \, w_{ms}(\alpha)$ for $j,i = 1,2,3$ $m,s = 1,2,\ldots,n$.

The range of summation over $\{jm\}$ is clear from the explicit forms of $\mathcal{A}$ and $\mathcal{D}$. The inversion of the matrix $\mathcal{D}$ is trivial, so we should now determine $\mathcal{A}^{-1}$. The matrix $\mathcal{A}^{-1}$ should be of the block form

$(\mathcal{A}^{-1})_{is,[\,]i} \rightarrow$

$(B_1)_{is,\ell k}$	$(B_2)_{is,\ell k}$	$(\tilde{B}_1)_{is,\sigma}$
$(B_3)_{is,\ell k}$	$(B_4)_{is,\ell k}$	$(\tilde{B}_2)_{is,\sigma}$
$\overline{(B_2)_{is,\ell k}}$	$\overline{(B_1)_{is,\ell k}}$	$\overline{(\tilde{B}_1)_{is,\sigma}}$
$\overline{(B_4)_{is,\ell k}}$	$\overline{(B_3)_{is,\ell k}}$	$\overline{(\tilde{B}_2)_{is,\sigma}}$

where $(B_1)_{is,\ell k}$ and $(B_2)_{is,\ell k}$ are defined for $i,s,\ell,k = 1,2,3$; $(B_3)_{is,\ell k}$ and $(B_4)_{is,\ell k}$ for $i,\ell,k = 1,2,3$ $s = 4,5,\ldots,n$; $(\tilde{B}_1)_{is,\sigma}$ for $i,s = 1,2,3$ $\sigma = 1,2,\ldots,6n-18$ and $(\tilde{B}_2)_{is,\sigma}$ for $i = 1,2,3$ $s = 4,5,\ldots,n$ $\sigma = 1,2,\ldots,6n-18$.

Demanding that $\mathcal{A}\mathcal{A}^{-1} = I$, we get the following equations

(1) $A B_1 = I$; (2) $A B_2 = 0$; (3) $A \tilde{B}_1 = 0$; (4) $\hat{A}_1 B_1 + \hat{A}_2 B_3 + \overline{\hat{A}_1}\,\overline{B_2} + \overline{\hat{A}_2}\,\overline{B_4} =$

(5) $\tilde{A}_1 \tilde{B}_1 + \tilde{A}_2 \tilde{B}_2 + \overline{(\tilde{A}_1 \tilde{B}_1 + \tilde{A}_2 \tilde{B}_2)} = I$.

From (1) one get $(B_1)_{is,\ell k} = \delta_{sk}\, x'_{\ell i}$, from (2) is $B_2 \equiv 0$ and from (3) $\tilde{B}_1 \equiv 0$. So we are left with two relations. The first one,

$$(5') \quad \sum_{j=1}^{3} \sum_{m=4}^{n} \left[{}^{L}\mathcal{H}_{\sigma,jm}(\alpha) (\tilde{B}_2)_{jm,\sigma'} + \overline{{}^{L}\mathcal{H}_{\sigma,jm}(\alpha)}\; \overline{(\tilde{B}_2)_{jm,\sigma'}} \right] = \delta_{\sigma\sigma'}$$

where using (I.3.11) and (I.2.3) one gets $(\tilde{B}_2)_{jm,\sigma'} = {}^{L}\Theta_{jm,\sigma'}(\alpha)$. The second one

$$(4') \quad \sum_{j=1}^{3} \sum_{m=1}^{3} {}^{L}\mathcal{H}_{\sigma,jm}(\alpha)\, \delta_{mk}\, x'_{\ell j} + \sum_{j=1}^{3} \sum_{m=4}^{n} {}^{L}\mathcal{H}_{\sigma,jm}(\alpha)\, (B_3)_{jm,\ell k} +$$

$$+ \sum_{j=4}^{n} \sum_{m=1}^{3} {}^{L}\mathcal{H}_{\sigma,jm}(\alpha)\, (-\overline{B_4})_{mj,\ell k} = 0$$

can be written with the help of the new matrix

$$M_{jm,\ell k} := \begin{cases} (B_3)_{jm,\ell k} & \text{for } j,\ell,k = 1,2,3 \quad m = 4,5,\ldots,n \\ \overline{(-B_4)_{mj,\ell k}} & \text{for } m,\ell,k = 1,2,3 \quad j = 4,5,\ldots,n. \end{cases}$$

Using (I.2.3) for $\delta \equiv 0$, one gets

$$M_{jm,\ell k} = \sum_{p=1}^{3} {}^{L}\Theta_{jm,pk}(\alpha)\, x'_{\ell p} \,. \quad \square$$

V.3. The generators of the unitary groups expressed through the new coordinates

The generators for the left $\mathcal{U}(3)$-action are given, as determined in (IV.3), by

$$ {}^L\chi_{iq} = {}^L\mathcal{C}_{iq}(x) - \overline{{}^L\mathcal{C}_{qi}(x)}, $$

where $ {}^L\mathcal{C}_{iq}(x) := -\sum_{s=1}^{n} x_{qs} \frac{\partial}{\partial x_{is}} $.

Let us now express these generators in the new $(x', \bar{x}', \alpha)$ coordinates.

V.3.1 Proposition: The left action generators of $\mathcal{U}(3)$ expressed through the new $(x', \bar{x}', \alpha)$ coordinates are

$$ {}^L\chi_{iq} = -\sum_{k=1}^{3} \left(x'_{qk} \frac{\partial}{\partial x'_{ik}} - \bar{x}'_{ik} \frac{\partial}{\partial \bar{x}'_{qk}} \right) $$

where the x'_{ik} are the parameters of $\mathcal{GL}(3, \mathbb{C})$.

Proof:

$$ \begin{aligned} {}^L\chi_{iq} = -\sum_{s=1}^{n} \Bigg\{ & \sum_{\ell=1}^{3} \sum_{k=1}^{3} \left(x_{qs} \frac{\partial x'_{\ell k}}{\partial x_{is}} - \bar{x}_{is} \frac{\partial x'_{\ell k}}{\partial \bar{x}_{qs}} \right) \frac{\partial}{\partial x'_{\ell k}} \\ & + \sum_{\ell=1}^{3} \sum_{k=1}^{3} \left(x_{qs} \frac{\partial \bar{x}'_{\ell k}}{\partial x_{is}} - \bar{x}_{is} \frac{\partial \bar{x}'_{\ell k}}{\partial \bar{x}_{qs}} \right) \frac{\partial}{\partial \bar{x}'_{\ell k}} \\ & + \sum_{\sigma=1}^{6n-18} \left(x_{qs} \frac{\partial \alpha_\sigma}{\partial x_{is}} - \bar{x}_{is} \frac{\partial \alpha_\sigma}{\partial \bar{x}_{qs}} \right) \frac{\partial}{\partial \alpha_\sigma} \Bigg\}. \end{aligned} $$

Using the explicit expressions for L^{-1} given in (V.2.2) and the relation $\Theta_{i\ell,km} = \Theta_{\ell i,mk}$ which is similar to (I.3.11), after a long but straightforward calculation, one gets the result. □

The generators for the right U(n)-action are given, as determined in (IV.3), by

$$ {}^R X_{ts} = {}^R C_{ts}(x) - \overline{{}^R C_{st}(x)}, $$

where

$$ {}^R C_{ts}(x) := \sum_{i=1}^{3} x_{it} \frac{\partial}{\partial x_{is}} . $$

We now express them in the new coordinates.

V.4.2 Proposition: The right action generators of U(n) expressed through the new coordinates $(x', \bar{x}', \alpha)$ are

$$ {}^{R}X_{ts} = \sum_{\sigma=1}^{6n-18} {}^{R}\Theta_{ts,\sigma}(\alpha)\frac{\partial}{\partial\alpha_\sigma} + \sum_{p=1}^{3}\sum_{m=1}^{3} {}^{R}\Theta_{ts,pm}(\alpha)\sum_{\ell=1}^{3}\left(x'_{\ell p}\frac{\partial}{\partial x'_{\ell m}} - \bar{x}'_{\ell m}\frac{\partial}{\partial \bar{x}'_{\ell p}}\right) $$

where the x''s are the parameters of $\mathcal{GL}(3,\mathbb{C})$ and α's are the parameters of the right coset space $U(n)/U(3)\times U(n-3)$.

Proof:

$$ \begin{aligned} {}^{R}X_{ts} = \sum_{i=1}^{3}\Bigg\{ &\sum_{\ell=1}^{3}\sum_{m=1}^{3}\left(x_{it}\frac{\partial x'_{\ell m}}{\partial x_{is}} - \bar{x}_{is}\frac{\partial x'_{\ell m}}{\partial \bar{x}_{it}}\right)\frac{\partial}{\partial x'_{\ell m}} \\ &+ \sum_{\ell=1}^{3}\sum_{m=1}^{3}\left(x_{it}\frac{\partial \bar{x}'_{\ell m}}{\partial x_{is}} - \bar{x}_{is}\frac{\partial \bar{x}'_{\ell m}}{\partial \bar{x}_{it}}\right)\frac{\partial}{\partial \bar{x}'_{\ell m}} \\ &+ \sum_{\sigma=1}^{6n-18}\left(x_{it}\frac{\partial\alpha_\sigma}{\partial x_{is}} - \bar{x}_{is}\frac{\partial\alpha_\sigma}{\partial \bar{x}_{it}}\right)\frac{\partial}{\partial\alpha_\sigma}\Bigg\}. \end{aligned} $$

To get the result, we again use the expression for L^{-1} given in (V.2.2) and also relations $\Theta_{i\ell,km} = \Theta_{\ell i,mk}$, then (I.2.2), (I.1.11) and (I.3.13). □

V.4. The Bargmann measure expressed through the new coordinates

The measure on the Bargmann space of square integrable analytic functions $F(x)$ is given by the expression

$$ d\mu(x) = \pi^{-3n}\prod_{j=1}^{3n}\exp(-\bar{x}_j x_j)\, d\,\mathrm{Re}\,x_j\, d\,\mathrm{Im}\,x_j \qquad 49 $$

where

$$ x_j = \frac{1}{\sqrt{2}}(\xi_j - i\pi_j) \equiv a_j + ib_j\,, \quad x_j \leftrightarrow x_{is} \quad i=1,2,3 $$

$s = 1,2,\ldots,n \quad j = 1,2,\ldots,3n$, so

$$ d\mu(x) = \pi^{-3n}\prod_{j=1}^{3n}\exp(-\bar{x}_j x_j)\, da_j\, db_j\,. $$

If one goes over to the new coordinates

$$(a_{(is)}, b_{(is)} \mid i=1,2,3 \;\; s=1,2,\ldots,n) \to (a'_{(ie)}, b'_{(ie)}, \alpha_\sigma \mid i,\ell=1,2,3 \;\; \sigma=1,2,\ldots,6n-18)$$

then the expression for the measure is

$$d\mu = \pi^{-3n} \exp\Big(-\sum_{i,\ell=1}^{3} \overline{x'_{ie}}\, x'_{ie}\Big) \prod_{i,\ell=1}^{3} da'_{(ie)}\, db'_{(ie)} \det \mathcal{J} \prod_{\sigma=1}^{6n-18} d\alpha_\sigma .$$

In order to determine this measure, one should calculate the Jacobian. The Jacobi matrix for our transition is given by (V.2.1)

$$\mathcal{J} = NLC$$

and
$$\det \mathcal{J} = \left[\det(\mathcal{J}^{+}\mathcal{J})\right]^{\frac{1}{2}}$$

$$= \left(\tfrac{1}{2}\right)^{3n-9} \left[\det(LL^{+})\right]^{\frac{1}{2}}$$

using the fact that $\mathcal{J}$ is real, and from the explict form of the matrices C and N.
The matrix LL^{+} can be easily determined in a block form

$$LL^{+} \to \begin{bmatrix} \left(\frac{\partial x}{\partial x'}\right)_{j\ell,is} & 0 & \left(\frac{\partial x}{\partial \alpha}\right)_{is,\sigma'} \\ 0 & \left(\frac{\partial \bar{x}}{\partial \bar{x}'}\right)_{j\ell,is} & \left(\frac{\partial \bar{x}}{\partial \alpha}\right)_{is,\sigma'} \\ \left(\frac{\partial x}{\partial \alpha}\right)_{\sigma,is} & \left(\frac{\partial \bar{x}}{\partial \alpha}\right)_{\sigma,is} & \end{bmatrix}$$

$$\begin{bmatrix} {}^{t}\overline{\left(\frac{\partial x}{\partial x'}\right)_{is,km}} & 0 & {}^{t}\overline{\left(\frac{\partial x}{\partial \alpha}\right)_{is,\sigma'}} \\ 0 & {}^{t}\overline{\left(\frac{\partial \bar{x}}{\partial \bar{x}'}\right)_{is,km}} & {}^{t}\overline{\left(\frac{\partial \bar{x}}{\partial \alpha}\right)_{is,\sigma'}} \end{bmatrix}$$

and its blocks are calculated using the explicit expressions for the matrix blocks of L from (V.2.1)

$$(LL^+)_{j\ell,km} = \delta_{j\ell,km} \equiv \delta_{jk}\,\delta_{\ell m}$$

$$(LL^+)_{j\ell,\tilde{k}\tilde{m}} = 0$$

$$(LL^+)_{\tilde{j}\tilde{\ell},km} = 0$$

$$(LL^+)_{\tilde{j}\tilde{\ell},\tilde{k}\tilde{m}} = \delta_{\tilde{j}\tilde{\ell},\tilde{k}\tilde{m}}$$

$$(LL^+)_{j\ell,\sigma'} = \sum_{\nu=1}^{3} \overline{x'_{j\nu}}\; {}^t\overline{\left({}^L\mathcal{F}\right)_{\nu\ell,\sigma'}(\alpha)}$$

$$(LL^+)_{\tilde{j}\tilde{\ell},\sigma'} = \sum_{\nu=1}^{3} x'_{\tilde{j}\nu}\; {}^t\left({}^L\mathcal{F}\right)_{\nu\tilde{\ell},\sigma'}(\alpha)$$

$$(LL^+)_{\sigma,km} = \sum_{\nu=1}^{3} x'_{k\nu}\; {}^L\mathcal{F}_{\sigma,\nu m}(\alpha)$$

$$(LL^+)_{\sigma,\tilde{k}\tilde{m}} = \sum_{\nu=1}^{3} \overline{x'_{\tilde{k}\nu}}\; \overline{{}^L\mathcal{F}_{\sigma,\nu m}(\alpha)}$$

$$(LL^+)_{\sigma\sigma'} = \sum_{i,\nu,w=1}^{3} \sum_{z=1}^{n} x'_{i\nu}\; {}^L\mathcal{F}_{\sigma,\nu z}(\alpha)\, \overline{x'_{iw}}\; {}^t\overline{\left({}^L\mathcal{F}\right)_{wz,\sigma'}(\alpha)} \quad + \text{c. c.}$$

The indices $\tilde{j}, \tilde{\ell}, \tilde{k}$ and $\tilde{m}$ are introduced in order to have the exact notation for the matrix block of LL^+.

Using the techniques for calculating the determinant of a matrix block form, one can show that the problem reduces to

$$\det(LL^+) = \det\left\{ \sum_{\sigma'} \sum_{\nu z} \sum_{wt} \sum_{\sigma''} {}^L\mathcal{F}_{\sigma\sigma''}(\alpha)\, P_{\sigma'',\nu z}\, T_{\nu z,wt}\, {}^t\overline{P_{wt,\sigma'''}}\; {}^t\overline{\left({}^L\mathcal{F}\right)_{\sigma'''\sigma'}(\alpha)} \right.$$

where P is given in (V.1.3) according to (I.3.2) and (I.3.3), T is as following

$$T_{\nu z,wt} = \begin{cases} \sum_{i=1}^{3} x'_{i\nu}\,\overline{x'_{iw}}\,\delta_{zt} & \text{for } \nu=1,2,3 \;\; z=4,5,\dots,n \;\; w=1,2,3 \;\; t=4,5,\dots,n \\ 0 & \text{for } \nu=1,2,3 \;\; z=4,5,\dots,n \;\; w=4,5,\dots,n \;\; t=1,2,3 \\ 0 & \text{for } \nu=4,5,\dots,n \;\; z=1,2,3 \;\; w=1,2,3 \;\; t=4,5,\dots,n \\ \sum_{i=1}^{3} \overline{x'_{iz}}\,x'_{it}\,\delta_{\nu w} & \text{for } \nu=4,5,\dots,n \;\; z=1,2,3 \;\; w=4,5,\dots,n \;\; t=1,2,3 . \end{cases}$$

Finally, one finds

$$\det(LL^+) = |\det {}^{L}\mathcal{F}_{\sigma\sigma'}(\alpha)|^2 \det T\, |\det P_{\sigma'',\nu z}|^2, \qquad \text{where}$$

$$\det T = \left(\det x' \,\overline{\det x'}\right)^{2(n-3)}, \qquad |\det P_{\sigma'',\nu z}| = 2^{3n-9}$$

so that

$$\det \mathcal{J} = \det\left({}^{L}\mathcal{F}_{\sigma\sigma'}(\alpha)\right)\left(\det x' \,\overline{\det x'}\right)^{n-3}$$

The measure is now

$$d\mu = \pi^{-(3n-9)}\left(\det x' \,\overline{\det x'}\right)^{n-3} d\mu(x')\; dm(\alpha), \qquad \text{where}$$

$$d\mu(x') = \pi^{-9}\, \exp\left(-\sum_{i,l=1}^{3} \overline{x'_{il}}\, x'_{il}\right) \prod_{i,l=1}^{3} da'_{(il)}\, db'_{(il)}$$

represents the exact Bargmann measure on the "reduced" space, depending only on x', and

$$dm(\alpha) = \det {}^{L}\mathcal{F}_{\sigma'\sigma''}(\alpha) \prod_{\sigma=1}^{6n-18} d\alpha_\sigma$$

is the proper measure on the right coset space U(n)/U(3)xU(n-3) according to (I.2.4) and from the fact that $\det d(\delta) = 1$ for U(j) [38].

V.5. The complete orthonormal basis in the new coordinates

Equipped with the measure on the Bargmann space of analytic functions which entirely factorize, we look for the set of basis functions which factorize in the corresponding way. Let us introduce the double Gelfand polynomials[27,56] of degree $N = \sum_{i=1}^{3} h_i$

$P^h_{\tilde{q}q}(x') = \lambda_h^{-1} D^h_{\tilde{q}q}(x')$, $D^h(x')$ is the nonunitary finite-dimensional irrep of $\mathcal{GL}(3,\mathbb{C})$ and $h=[h_1h_2h_3]$. They are orthonormal with respect to the inner product of Bargmann[56]

$$\int \overline{P^{h'}_{\tilde{q}'q'}(x')}\, P^h_{\tilde{q}q}(x')\, d\mu(x') = \delta(h',h)\,\delta(\tilde{q}',\tilde{q})\,\delta(q',q).$$

V.5.1 Proposition: The double Gelfand polynomials are still orthogonal with respect to

$$d\sigma(x') := \left(\Delta^{123}_{123}\ \overline{\Delta^{123}_{123}}\right)^{n-3} d\mu(x') \quad , \text{ where}$$

$\Delta^{123}_{123} \equiv \det x'$, and can be normalized

$$\tilde{P}^h_{\tilde{q}q}(x') := \frac{\lambda_h}{\lambda_{h+n-3}} P^h_{\tilde{q}q}(x'),$$

where

$$\lambda_h = \left[\frac{(h_1+2)!\,(h_2+1)!\,h_3!}{(h_1-h_2+1)(h_1-h_3+2)(h_2-h_3+1)}\right]^{\frac{1}{2}}.$$

Proof: Note that

$$D^{[h_1h_2h_3]}_{\tilde{q}q}(x') = D^{[h_1-h_3\ h_2-h_3\ 0]}_{\tilde{q}-h_3\ q-h_3}(x')\left(\Delta^{123}_{123}\right)^{h_3}$$

where $\tilde{q}-h_3$ means that all Gelfand pattern indices $\tilde{q}_{ij}$ go over into $\tilde{q}_{ij}-h_3$. Hence

$$D^h_{\tilde{q}q}(x')\left(\Delta^{123}_{123}\right)^{n-3} = D^{h+n-3}_{\tilde{q}+n-3\ q+n-3}(x'),$$

where $h+n-3 \equiv [h_1+n-3\ \ h_2+n-3\ \ h_3+n-3]$.

Using that relation one easily sees that

$$\int \overline{P^{h'}_{\tilde{q}'q'}(x')}\; P^{h}_{\tilde{q}q}(x')\, d\sigma(x') =$$

$$= \int \overline{\lambda^{-1}_{h'}\, D^{h'+n-3}_{\tilde{q}'+n-3\; q'+n-3}(x')}\; \lambda^{-1}_{h}\, D^{h+n-3}_{\tilde{q}+n-3\; q+n-3}(x')\, d\mu(x')$$

$$= \frac{\lambda^{2}_{h+n-3}}{\lambda^{2}_{h}}\, \delta(h',h)\, \delta(\tilde{q}',\tilde{q})\, \delta(q',q)\,. \quad \square$$

V.5.2 Proposition: Let G be a compact Lie group with a subgroup H. Consider an irrep D^f of G which subduces the irrep D^r of H. Denote the rows and columns of D^r by indices $i = i(r)$, $i' = i'(r)$ and the rows of D^f by the pairs $r\; i(r)$. Writing a general element $g \in G$ in the form $g = hc$ where $h \in H$, $c \in G/H$ one finds

$$D^{f}_{r\,i(r),j}(hc) = \sum_{i'(r)} D^{r}_{i(r)\,i'(r)}(h)\, D^{f}_{r\,i'(r),j}(c)$$

where the matrix elements $D^{f}_{r\,i(r),j}(c)$ fulfil the orthogonality relation

$$\sum_{i(r)} \int \overline{D^{f'}_{r\,i(r),j'}(c)}\; D^{f}_{r\,i(r),j}(c)\, dm(c)$$

$$= \delta(f',f)\, \delta(j',j)\, (|G|/|H|)\, (\dim(H,r)/\dim(G,f))\,.$$

Proof: The general orthogonality relation for unitary representations of G can be decomposed according to the factorization $g = hc$. The result is obtained making use of (I.2.4). $\square$

V.5.3 Proposition: Consider the partitions $h \equiv [h_1 h_2 h_3]$ with $h_1 + h_2 + h_3 = N = 0,1,2,\ldots$ Denote by $\tilde{q}$ and q Gelfand patterns for the representation h of $\mathcal{U}(3)$ and U(n), respectively. The double

Gelfand polynomials

$P^h_{\tilde{q}q}(x)$

form a complete orthonormal basis of state space with the Bargmann measure $d\mu(x)$. This basis transforms irreducibly under the action of the groups $\mathcal{U}(3)$ and U(n). □

V.5.4 Proposition: The states

$$\tilde{P}^h_{\tilde{q}q}(x', w(\alpha)) = \pi^{\frac{3n-9}{2}} \sum_{\tilde{q}'} \tilde{P}^h_{\tilde{q}\tilde{q}'}(x') \, \tilde{D}^h_{\tilde{q}'q}(w(\alpha))$$

for $\tilde{q}, \tilde{q}'$ being Gelfand patterns of the irrep $h \equiv [h_1 h_2 h_3]$ of $\mathcal{U}(3)$ and with

$$\tilde{D}^h_{\tilde{q}'q}(w(\alpha)) = \left[\frac{\dim(n,h)}{\dim(3,h)}\right]^{\frac{1}{2}} \left[\frac{|U(3)|\,|U(n-3)|}{|U(n)|}\right]^{\frac{1}{2}} D^h_{\tilde{q}'q}(w(\alpha))$$

form a complete orthonormal basis of state space with the new measure

$$\pi^{-3n+9}\, d\sigma(x')\, dm(\alpha).$$

Proof: By using the orthonormality of the polynomials $\tilde{P}^h(x')$ given in proposition V.5.1 one finds

$$\left(\tilde{P}^{h'}_{\tilde{q}'q'}, \tilde{P}^h_{\tilde{q}q}\right) =$$

$$= \delta(h',h)\,\delta(\tilde{q}',\tilde{q})\,\frac{\dim(n,h)}{\dim(3,h)}\,\frac{|U(3)|\,|U(n-3)|}{|U(n)|} \times$$

$$\times \sum_{\tilde{q}''} \int \overline{D^h_{\tilde{q}'q'}(w(\alpha))}\, D^h_{\tilde{q}''q}(w(\alpha))\, dm(\alpha).$$

With respect to the irrep D^h of U(n), the label $\tilde{q}''$ may be interpreted as a Gelfand scheme with all weight components $w_i = 0$ for $3 < i \leq n$ so that the subduced representation of U(3)xU(n-3) is $D^h \times D^{[0]}$.

Application of proposition V.5.2 to the integral over the coset U(n)/U(3)xU(n-3) then yields

$$\left(\tilde{P}^{h'}_{\tilde{q}'q'}, \tilde{P}^{h}_{\tilde{q}q}\right) = \delta(h',h)\,\delta(\tilde{q}',\tilde{q})\,\delta(q',q).$$

The completeness of the new basis follows from proposition V.5.5. □

V.5.5 Proposition: The basis of state space given in propositions V.5.3 and V.5.4 are explicitly connected by the transformation of coordinates $x = x' w(\alpha)$, that is

$$P^{h}_{\tilde{q}q}(x'w(\alpha)) = \pi^{\frac{3n-9}{2}} \sum_{\tilde{q}''} \tilde{P}^{h}_{\tilde{q}\tilde{q}''}(x')\,\tilde{D}^{h}_{\tilde{q}''q}(w(\alpha)).$$

Proof: Upon writing $x = x' w(\alpha)$ and using the transformation properties of the double Gelfand polynomials of proposition V.5.3, these states may be rewritten as

$$P^{h}_{\tilde{q}q}(x'w(\alpha)) = \sum_{\tilde{q}''} P^{h}_{\tilde{q}\tilde{q}''}(x')\,D^{h}_{\tilde{q}''q}(w(\alpha)).$$ [56]

Since x' is a 3 x 3 matrix, the label $\tilde{q}''$ runs only over those Gelfand patterns which correspond to the irrep $h \equiv [h_1, h_2, h_3]$ of U(3). Using the explicit expressions for λ_h and $\dim(j,h)$ [56] one finds

$$\frac{\lambda_{h+n-3}}{\lambda_h} = \left[\frac{\dim(n,h)}{\dim(3,h)}\right]^{\frac{1}{2}}$$

For the volume one finds [36]

$$\left[\frac{|U(3)|\,|U(n-3)|}{|U(n)|}\right]^{\frac{1}{2}} = \pi^{-\frac{3n-9}{2}}$$

Therefore the expressions P^h and D^h may be converted into the expressions $\tilde{P}^h$ and $\tilde{D}^h$ and yield the result. □

The implication of propositions V.5.4 and V.5.5 for the analysis of an U(n)-invariant collective Hamiltonian are as follows: The collective Hamiltonian must be some function of the generators of the group $\mathcal{U}(3)$ or $g\mathcal{L}(3,\mathbb{C})$ and its eigenstates must be of the form

$$\sum_{\tilde{q}} c_{\tilde{q}} \tilde{P}^{h}_{\tilde{q}q}(x', w(\alpha))$$

where the coefficients depend only on $\tilde{q}$ for fixed h. Since the Hamiltonian does not affect the coordinates α and the index q, one can make the special choice $q = q_{max}$ and $\alpha = 0$ Then the states become

$$\tilde{P}_{\tilde{q}\, q_{max}}(x', w(0)) = b_h \tilde{P}^{h}_{\tilde{q}\, q_{max}}(x')$$

where the polynomials on the right-hand side are defined in proposition V.5.1 and the constant b_h can be obtained from proposition V.5.4. The determination of the coefficients $c_{\tilde{q}}$ can now be done on the collective state space spanned by the polynomials $\tilde{P}^{h}(x')$ with the measure $d\sigma(x')$.

V.5.6 Proposition: The eigenstates and eigenvalues of an U(n)-invariant collective Hamiltonian can be obtained on the collective state space with the basis $\tilde{P}^{h}(x')$ and the measure $d\sigma(x')$.

Summary and Conclusion

In this thesis some problems related to the collective motion in nuclear physics have been studied. These problems were considered using the concepts of collective kinematical and intrinsic symmetry groups which in chapter II and III led to the coordinate transformation in configuration space. This analysis were related to the dynamical group MQC (see p. 10). In the phase space we introduced the collective $\mathcal{U}(3)$ group both as a kinematical and a dynamical group and U(n) as a symmetry group for the collective motion (chapter IV). The collective kinematical group need not be identical to the dynamical group: in chapter V we introduce the coordinate transformation induced by the $\mathcal{GL}(3,\mathbb{C})\times U(n)$ group.

In chapter I we gave some mathematical preliminaries referred to the action of a group on different types of manifolds such as group manifolds and coset spaces (homogeneous spaces). On all these manifolds the group act transitively. Such problems have been studied extensively[37] in the mathematical literature. In our particular problems, studied in chapters II to V, the manifold on which we defined the action is much more complicated: the configuration space or phase space for our problem was defined as some cartesian product of manifolds. When we define some action of a certain group $\tilde{G}$ on such a manifold it often happens that the action does not stay on one of the factors of the cartesian product, but instead affects for example two of them: in chapter III the SO(n) right action on the 3n-dimensional manifolds acts both on the right coset space SO(n)/SO(3)xSO(n-3) and on the neighboring manifold $SO(3)$ on which the vortex spin is defined. Connected with these problems we studied in particular the left and the right "action" of a group on different manifolds, necessary for example for the definition of differential operators in the body (see chapter II and III). Some differential operators in the body are no longer the generators of some action because they "act" on the coset spaces from the "wrong" side. We had not found these problems systematically studied in the mathematical literature, so we

devoted chapter I to this analysis. A particularly important result for this analysis is the proposition (I.2.3) which is related to some "intermediate", combined left and right "action" between two manifolds. The problems for which we needed this proposition also appear in phase space, with the difference that in phase space the place "between" two factor manifolds is not interpreted as the body fixed system. Although mathematically equivalent this place has not yet obtained a "physical" name. We solved the problem of finding the invariant volume element in each coordinate scheme in chapters II, IV and V, by use of the $\mathcal{U}$ matrices from propositon (I.2.4). Both propositions (I.2.3) and (I.2.4) are coordinate independent. We consider them as new results.

In chapter II we extensively studied the $SO(3)\times SO(n)$coordinate scheme in order to show how our method works on the already solved problem[21,22]. We also gave a criticism of the interpretation of the results related to the eventual total separation of kinetic energy into collective and intrinsic parts w.r.t. the $gl_+(3,\mathbb{R})$ collectivity[23,24,25,26]. We claimed that the authors obtained collective kinetic energy w.r.t. $gl_+(3,\mathbb{R})$ collectivity, but that the part which they considered as the totally separated intrinsic one actually contains the hidden coupling term. This happened in our opinion because the authors used the $SO(3)\times SO(n)$coordinate scheme, which is not adapted for the $gl_+(3,\mathbb{R})$collectivity.

In chapter III we developed for the first time the $gl_+(3,\mathbb{R})\times SO(n)$ scheme, the only one in which it makes sense to try to answer the questions related to the $gl_+(3,\mathbb{R})$collectivity. We studied the problem of separation of the kinetic energy and the velocity field. This study is important for the investigations which should show whether the MQC phenomenological model[17] can be related to some microscopic theory[24,25,26]. We conclude that the total separation of the collective kinetic energy and the velocity field w.r.t. $gl_+(3,\mathbb{R})$ collectivity is not possible in the frame of the coordinate transformation for n+1 particles. That is due to the coupling terms which appear. Out of the coupling terms in the kinetic energy and the

velocity field, which are all "weighted" with a certain matrix block $\Theta_{mk,\ell<j}(\beta) m,1,j = 1,2,3 \quad k = 4,5,\ldots,n \quad \beta \in SO(n)/SO(3)\times SO(n-3)$ we find out the condition under which the separation would be achieved: $\Theta_{mk,\ell<j}(\beta) \equiv 0$. This condition does not follow from the theory of Lie groups. Note that our result comes as the indirect consequence of proposition (I.2.3).

In chapter IV we introduced in phase space the coordinate scheme related and adapted to the ansatz of U(n) invariance for the collective hamiltonian. For this scheme we calculated the Jacobi matrix, its inverse, the left and right action generators for the $\mathcal{U}(3)$ and U(n) groups respectively, and the invariant volume element in the new coordinates. These studies have their sources in ref. 27,28,35 and also 5 and are related to ref. 57 .

In chapter V we introduced in phase space the $g\mathcal{L}(3,\mathbb{C})\times U(n)$ scheme which has only one orbit unifying all different types of orbits of the $\mathcal{U}(3)\times U(n)$ scheme studied in chapter IV and in ref. 35. Also, both concepts of collectivity considered in chapter IV are unified in this scheme. In this scheme a completely orthonormal basis is constructed. It consists of sums of products of collective and intrinsic states. The collective states are polynomials in a new Bargmann space of analytic functions in nine complex variables. The intrinsic states are matrix elements of irreducible representations of U(n) on a coset space of this group. An U(n)-invariant collective hamiltonian can be studied on the collective state space of Bargmann type.

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LEBENSLAUF

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Ich bin im Jahre 1949 in Beograd geboren. Von 1957 bis 1964 besuchte ich die Grundschule und danach bis 1968 das Gymnasium in Beograd. 1968 begann ich mit dem Studium der Physik an der Naturwissenschaftlich-Mathematischen Fakultät in Beograd. Die Diplomprüfung legte ich 1973 mit der Arbeit "Viriale Zustandsgleichung" bei Prof. Z. Marić ab. Im Jahre 1974 erhielt ich eine Anstellung am Institut für Physik in Beograd.

Unter Anleitung von Prof. F. Herbut schloß ich 1977 meine Magisterarbeit mit dem Titel "Linear-antilineare Darstellungen der Symmetriegruppe des Hamiltonschen Operators und kanonischen Basen" ab. Daraus entstand die folgende Publikation: Herbut, F., Vujičić, M. and Papadopolos, Z., J. Phys. A13 (1980) 2577.

Im Oktober 1978 kam ich als DAAD-Stipendiatin an die Universität Tübingen zu Prof. Kramer, unter dessen Betreuung ich wissenschaftlich arbeitete. Im Januar 1980 wurde ich als Doktorandin an der Fakultät für Physik aufgenommen. Nach Ablauf des DAAD-Stipendiums erhielt ich im Februar 1980 eine Stelle als wissenschaftliche Hilfskraft am Institut für Theoretische Physik; seit Januar 1983 bin ich wissenschaftliche Angestellte im Rahmen des DFG-Projekts "Symmetrie physikalischer Vielteilchensysteme".

In Zusammenarbeit mit Prof. Kramer entstanden die folgenden Publikationen:

1) Kramer, P. und Papadopolos, Z., "Collective Motion, Composite Particle Structure, and Symplectic Groups in Nuclei" in: Lecture Notes in Physics 135, ed. K.B. Wolf (Springer, Berlin 1980)

2) Papadopolos, Z. and Kramer, P., Physica 114A (1982) 309

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5) Kramer, P., Saraceno, M., Papadopolos, Z. and Schweizer, W., "Towards a geometric theory of collective motion" in: Proceedings of the International Conference on Nuclear Physics Vol. 1, Contributed Papers, Bologna 1983

6) Papadopolos, Z. and Kramer, P., "Is it possible to separate the kinetic energy and the velocity field into a collective and an intrinsic part with respect to the $\mathfrak{gl}_+(3, \mathbb{R})$ collectivity?", to be published in the Proceedings of the XII International Colloquium on Group Theoretical Methods in Physics, Trieste 1983

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